VIBRATION AND BENDING BEHAVIOR OF LAMINATED COMPOSITE PLATE WITH UNCERTAIN MATERIAL PROPERTIES USING FUZZY FINITE ELEMENT METHOD

by

RAVI PRATAP SINGH

DEPARTMENT OF MECHANICAL ENGINEERING

National Institute of Technology Rourkela

JUNE, 2015

Vibration and Bending Behavior of Laminated Composite Plate with Uncertain Material Properties Using Fuzzy Finite Element Method

Thesis Submitted to

National Institute of Technology, Rourkela

For the award of the degree
of

Master of Technology
In Mechanical Engineering with Specialization
"Machine Design and Analysis"

by

Ravi Pratap Singh
Roll No. 213me1394

Under the Supervision of
Prof. Subrata Kumar Panda

DEPARTMENT OF MECHANICAL ENGINEERING
National Institute of Technology Rourkela

JUNE, 2015

DEPARTMENT OF MECHANICAL ENGINEERING

NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA

DECLARATION

I hereby declare that the work which is being presented in the thesis entitled "**Vibration and Bending Behavior of Laminated Composite Plate Using Fuzzy Finite Element Method**" for the award of the degree of Master of Technology in Mechanical Engineering, submitted in the Department of Mechanical Engineering, National Institute of Technology Rourkela, 769008, Odisha, India, is an authentic record of my own work carried out under the supervision of **Prof. (Dr.) S.K. Panda**.

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

Place: Rourkela

Date:

(RAVI PRATAP SINGH)

Roll No. 213ME1394

Department of Mechanical Engineering

National Institute of Technology, Rourkela

769008, Odisha, India

DEPARTMENT OF MECHANICAL ENGINEERING

NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA

CERTIFICATE

It is certified that the work contained in the thesis entitled "**Vibration and Bending Behavior of Laminated Composite Plate Using Fuzzy Finite Element Method**" by Ravi Pratap Singh, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

Place: Rourkela

Date:

(Dr. S. K. PANDA)
Assistant Professor,
Department of Mechanical Engineering
National Institute of Technology Rourkela
769008, Odisha, India

Dedicated To

My Parents

ACKNOWLEDGEMENTS

First of all I express my recondite sense of gratitude and profound appreciation to my thesis supervisors **Prof. S.K. Panda**, for their inspiration and valuable suggestions in carrying out this work. I find myself fortunate to have worked under him for the freedom, constant encouragement he gave throughout this work and for having introduced me to the fascinating world of Uncertainty and Composite Materials. I sincerely admit that I owe him a lot and am ever indebted to him for all that he has done to me.

I am very much thankful to **Prof. D.R Parhi**, **Prof. S.C. Mohanty**, **Prof. J. Srinivas**, **Prof. R.K Behera** and **Prof. S.K. Behera**, for all their support and encouragement during the entire course of stay.

I sincerely thank **Prof. S.S Mohapatra**, Head of Department of Mechanical Engineering, and the other faculty of the department for providing moral supports.

I am indebted to my lab mates **Vishesh Ranjan Kar, Vijay Kumar Singh, Pankaj K.V. Katariya, Kulmani Mehar, Sushree Sasmita Sahoo, Deeprodyuti Sen, Ajay Kumar Paswan** and **Surajsitaram Bankar** for their help and support during my stay in laboratory and making it a memorable experience in my life.

My heartful thanks to **Mrs. Niharika Satapathy, Mrs. Padmaja Tripathy, Mrs. Khushboo Singh, Mrs. Chanda and Mrs. Senapati** for their hospitality and support and special thanks to **Google** and **Om** for providing me hearty entertainment.

I cannot forget **Mr. Sajal Kumar Chanda**, who always ready for every kind of help during my stay at NIT Rourkela.

Sincere thanks are due to my other friends **Himalaya, Prabhajyoti, Ashish, Nihar, Viswa Teja, Gaurav, Vishwajeet, Prajapati, Ashutosh** and all my batch mate for providing moral supports.

I do not have adequate words to express my indebtedness to my family members and relations for all their pains and suffering. I express my thanks to all of them for their patience, cooperation and understanding, during the course of this work.

I record my sincere apologies to those whose names I have inadvertently missed despite their meaningful contribution during the course of this work.

Ravi Pratap Singh

LIST OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

(X, Y, Z)	Cartesian coordinate axes
$(u_\alpha, v_\alpha, w_\alpha)$	Fuzzy displacements along X, Y and Z directions
$(u_{0\alpha}, v_{0\alpha}, w_{0\alpha})$	Fuzzy displacements of a point on the mid-plane of the panel along X,Y and Z direction
a, b, h	Length, breadth and thickness of plate
$\theta_{x\alpha}$, $\theta_{y\alpha}$	Fuzzy rotations with respect to y and x direction respectively
$\theta_{z\alpha}$, $\phi_{x\alpha}$, $\phi_{y\alpha}$, $\lambda_{x\alpha}$, $\lambda_{y\alpha}$	Fuzzy higher order terms of Taylor series expansion
$\{\mathcal{E}_\alpha\}$	Fuzzy strain vector
$\{\delta_\alpha\}$	Fuzzy displacement vector
E	Young's modulus
G	Shear modulus
ν	Poisson's ratio
$[K_\alpha]$	Fuzzy stiffness matrices
$[T]$, $[f]$	Function of thickness coordinate
U	Total Strain energy
V	Total Kinetic energy
ω	Frequency

ABSTRACT

In this work, uncertainty in material properties of the laminated composite plate such as young's modulus, shear modulus, and Poisson's ratio are modeled using fuzzy approach. A fuzzy-based finite element model is developed for the analysis of laminated composite plate involving fuzziness in the material properties. The mathematical model is developed using fuzzy arithmetic based on the higher-order shear deformation theory. Based on the proposed and developed model the vibration and bending responses of the laminated composite plate are being analyzed taking uncertainty in material properties. Applicability and feasibility of the methodology incorporated are presented with some numerical examples.

Keywords- ***Fuzzy finite element method, laminated composite plate, uncertain material properties, fuzzy arithmetic, higher order shear deformation.***

Chapter- I
INTRODUCTION

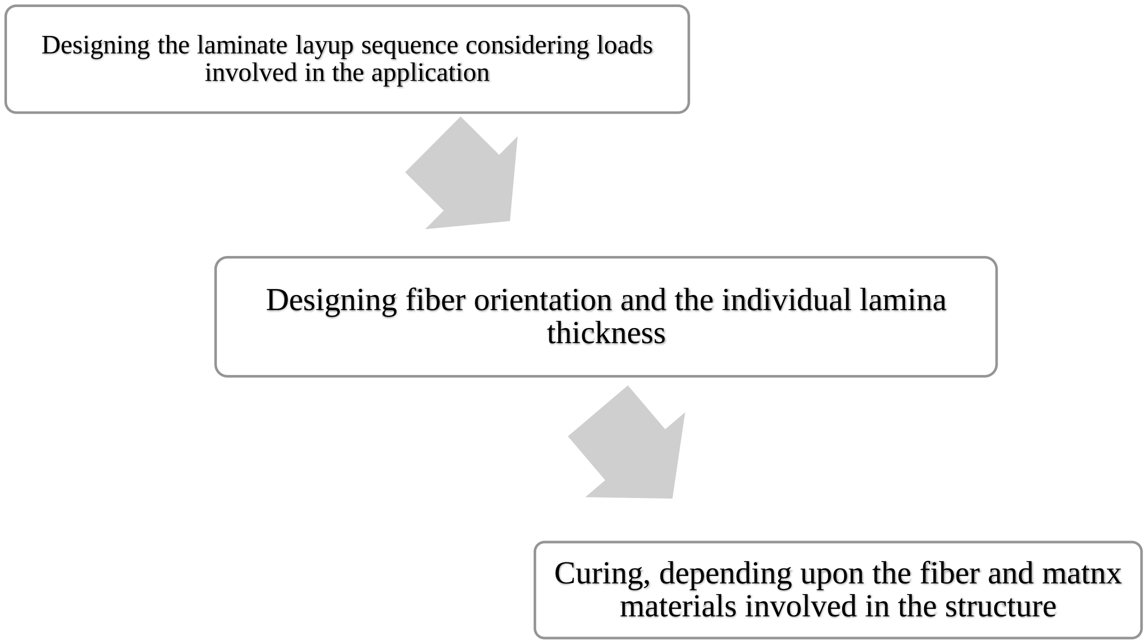
1.1 Overview

Composite materials may be defined as, that class of materials in which there are two or more chemically distinct constituents, having a different interface separating them. Polymer matrix composite material with the well-known advantages at nominal temperatures found to be extensively used in countless engineering structures. In view, that composites have high stiffness and strength to weight ratio, outstanding fatigue resistance, and corrosion resistance characteristics compared to conventional engineering materials hence, composite materials beat conventional structural materials in many areas.

The essential difference between composites and conventional structural materials is that composites materials are heterogeneous materials, though we treat them as homogeneous at the macro level. This is an added advantage to the designer in terms of optimization of structural response. So the design of the structure can be interlinked with the design of the composites used, to produce optimized structures with much superior performance than structures with conventional materials. Fiber reinforced composite materials have structurally superior quality fibers bonded together by a low-performance matrix material. Reinforcement material introduced in the composite in either discontinuous or continuous form. The basic building block for a composite structure is a lamina, which is very thin, and cannot be employed for withstanding external loads. For this reason, laminas are stacked together to make a laminate. Multi ply laminates employing continuous fiber composites are the most important class of composites for structural applications.

The fibers used in the composite are glass, carbon, graphite. Glass fibre composites are widely used and their processing technology is highly developed, because of low manufacturing cost and are used in structures like ship hulls, antennas, sports goods, etc. Carbon/graphite fibers have greater stiffness, superior fatigue strength, low coefficients of thermal expansion, better heat resistance and excellent performance under compressive as well as tensile loads, so they are used in the aerospace industry. In certain application, hybrid composites are used to take advantage of the greater properties of different types of fibers, which result in lowering the cost and weight as compared to structure made up of a single type of fiber only.

The manufacturing techniques used for composites are very much different from that of used for conventional materials like metals. Manufacturing of a continuous fiber reinforced composite, like a plate, involves following step:



Every stage in composite manufacturing involves processes which can often be difficult to control to a high accuracy resulting into variability in the material properties. The primary sources of this variability in composites material properties can be listed as follows:

1. Production-related fiber and matrix property variability,
2. Variations in intermediate materials (e.g. prepregs, sheet molding compounds);
3. Local and overall variations in fiber volume fraction,
4. Variations in fiber orientation resulting from various sources such as resin flow and poor initial placement, variations in fabrication processes and voids.

Based on the structure and the material used, a broad range of manufacturing processes like filament winding, pull trusion and hand layup are used for the manufacturing of composites. Along with various processing factors like a void fraction, the orientation of the fiber, fiber volume fraction and interfacial bond characteristics play a vital role in deciding the properties of the composite. Processing of fibers and the curing cycle of the composite determine fiber-matrix interface, and this determines the strength of the fiber matrix interface that plays a significant role in influencing the properties of the manufactured composite. Thus, owing to the inherent uncertainties involved in processing/ manufacturing techniques of composites, the final manufactured product can have substantial variations in the properties regarding the design values. The variability in characteristics of the fiber and the matrix materials add to the dispersion in the material properties of the end product.

1.2 System Modeling and Analysis of Composites Structures

Conventional methods for modeling and analysis of the structure involves various step like modeling the system by preassuming some basic parameter, boundary conditions and system characteristics, which are supposed to be deterministic in nature. Typically in the modeling stage, an empirical term named the factor of safety is introduced, and it assumed that the factor of safety covers the unknown risks associated with the actual property variability in the structure. However, in practice the system parameters are seldom deterministic. Thus, we can have a different class of problem modeling in which the various parameters involved in the analysis are either probabilistic or deterministic in nature or a combination of both.

When material properties parameters like density, modulus and poisson's ratio are considered uncertain, resultant response parameters like natural frequencies, deflections, stresses, strains, buckling loads are also varies because they are functions of basic random material property parameters. The performance characteristics of structure and risk of failure can be assessed accurately only by adopting a probabilistic analysis approach or fuzzy based analysis approach.

As composite materials are widely used in automobile, aerospace, marine structures and space projects, like space stations, advanced aircraft and helicopter. Computer based simulation of some of the complex configurations often shows, tightly packed/overlapping natural frequencies of some of its components. In such cases, even the smallest shift in characteristics of the components can have a definite effect on the response of the structure. Apart from randomness in material property parameters, there could be randomness in geometry, loading and boundary condition also.

The general solution techniques available for the uncertainty analysis in the composite structure and are:

1.2.1 Perturbation Techniques

In perturbation techniques, the random parameters are stated in the form of a truncated asymptotic expansion, by introducing a small parameter. Approximation of this kind is usable only if the randomness is much lower than the standard mean value of the random parameter. However, This condition is satisfied in the majority of the engineering problems like in the aerospace engineering. The approach can normally be accepted for the majority of practical engineering situations and which constructs the system equation that varies with the level of uncertainties in the random quantity. Advantages of this techniques are that, the random part and deterministic part of the solution can be separated from these expanded system equations, these two obtained solutions can be future combined to get the complete solution of the problem as the characteristics of the response of the system.

1.2.2 Monte Carlo Simulation

In monte carlo simulation (MCS) technique; first, a random sample of the system parameters is generated and then it is used to compute the response by repeated numerical trials. Statistics of the unknowns can be found from as the results of a vast number of such samples. Although, the sample size is the function of the nature of the problem considered and random variables (Stefanou, 2009). The very complex issues that are otherwise very challenging or difficult to solve using other techniques, now can be solved using MCS method. ompu The complex problem that was otherwise difficult to solve by using other techniques, now can be easily solved by the use of MCS method. Th main drawback of this method is that it requires too much time and computing resources.

1.2.3 Stochastic Finite Element Method

These random parameters are modeled using probability distribution such as gaussian and non-gaussian. The basic formulation techniques used to take into account the randomness in finite element equations are the MCS or perturbation method or a combination of these two. The significant advantage of this approach is that it makes possible to handle uncertainty in the realistic structures with the efficient treatment of the large-scale problems. In stochastic finite element method (SFEM), randomness in parameters is taken at the element level. The classical deterministic finite element approach is extended as SFEM for computing the static and dynamic problems involving uncertainty in material, geometric and loading parameters (Stefanou, 2009).

1.2.4 Fuzzy Finite Element Method

In fuzzy finite element method (FFEM), the objective is to determine the membership function of an output quantity, based on the fuzzy description of the input random system parameters (F. Massa, Tison, and Lallemand, 2009). The discretization of the membership function is required for computing fuzzy numbers. Initially, the discrete fuzzy numbers are obtained from cuts on the basis of the degree of membership level [0,1], for each α-cuts level the interval are defined by a lower bound and upper bounds. Then the membership functions of the uncertain system parameters and converted into a set of intervals using α-sublevel technique and the global matrix formulation are built for each α-cut level. After evaluating element matrices, assembled to generate system matrices that result as fuzzy equations. Fuzzy equations are solved by using an interval of confidence at finite α-cut (Rao and P., 1995), which reduced to a solving system of interval equations. External loads and boundary condition applied as per deterministic finite element analysis (FEA) procedure. The biggest advantage of this techniques is that there is no risk of loss of information, can handle linguistic design parameter as required and the computational cost is very low as compared to other method available methods discussed earlier.

1.3 Motivation of The Present Work

The laminated composite structures are of high focus to the structural designers because of its low thermal conductivity, dimensional stability, high-impact strength, and corrosion resistance. In the last few years, composite structures have been used extensively in aerospace/aeronautical engineering, which enforced the need for the precise and reliable analysis of composite structures. Meanwhile, a greater numbers of methods of structural analysis are introduced and underwent sea changes with the introduction of high-speed computing.

Nowadays the majority of the composite structural analysis are performed with the help of finite element method (FEM). In finite element analysis, a mathematical model is designed to predict accurately the behavior of the composite structure with the assistance of various parameters like material properties, geometric properties, support conditions and loads applied. Moreover, all these parameter influence the behavior of structures. The parameters that are being used to generate a mathematical model are usually certain and deterministic in nature. However in composite structures, the parameter like fiber volume fraction, elastic modulus and geometry are not know precisely. The information available might be incomplete, imprecise, linguistic or vague because of the compound process and involvement of human judgment. For example, composite manufacturing involves various parameters like cure temperature, resin chemistry, and pressure; which contributes to the final characteristics of the composite material. Although the process time and degree of cure can be determined only by the experiments. In the similar manner due to the involvement of machine tolerance, the geometric dimension and fiber orientation cannot be achieved accurately. Thus, the majority of the composite structure parameters involve uncertainty and imprecise, which are difficult to be controlled conveniently by using probabilistic and deterministic approaches.

If imprecise values of the composite material parameters are used to find the response of the structure, then the reliability and accuracy of the results are at the risk. On one hand, the probabilistic approach can be used for finding out the probability distributions of the uncertain parameters, which requires a significant amount of information, which in many cases not available and or imprecise. On the other hand extension of the FEM to FFEM with the help of fuzzy set theory and arithmetic can be conveniently used to account for uncertainties in composite structure parameters. Forming fuzzy finite element analysis of laminated composite plate will be beneficial for the analysis of composite structures.

1.4 Objective and Scope of Present Thesis

Recently, many researchers have tried to solve the problem of laminated composite plate with uncertainty in the material properties. Some of the used methods are computationally expensive and time consuming.

The present work is an attempt in terms of FFEM to fill the gaps. The broader objectives at the present research can be summarized as under:

1. Development of mathematical model of laminated composites plate with uncertain material properties for the analysis of the mechanical response (vibration and bending).
2. Development of FFEM based on the developed mathematical model.
3. Development of a customized computer code for the proposed mathematical model in MATLAB environment.
4. Fuzzification of material properties parameter i.e. Young’s modulus, Poisson’s ratio, Shear modulus etc.
5. Validation of proposed FFEM of composite plate involving uncertainty with published results.
6. Investigation of Influence of uncertain material properties on the mechanical behaviours of laminated composite plate.

1.5 Organization of the Thesis

The thesis comprises of six chapters; the first chapter presents a brief overview of the area of study. A short review of the relevant literature is available related with current work is presented in chapter two. Which can help to find out the knowledge gap or motivation behind the study and objective of the work. An brief description of the basic definition and details relevant to fuzzy computation and that applies to present work is presented in chapter three. General mathematical formulation for the FFEA is presented in the chapter four. Chapter five discuss about the validation of the proposed model and specific problems related to free vibration and static analysis of the laminated composite plates with the published literatures. Chapter six describes the general conclusions, contribution of the thesis and suggestions for the future research.

1.5 Summery

The composite structure parameter are often uncertain in natural comparison to those made of conventional engineering materials. Based on the available test data in the literature it appears that the real test values have a wide dispersion. Commonly, for conventional methods of the analysis of composites structures, all the parameters that describe the system are considered to be deterministic in the nature. Thus the results obtained, do not reflect the effects of the real life uncertainty of the parameters involved. Moreover, it may also predict the system behavior inaccurately. For more sensitive applications like aerospace structure, the problem of analysis of composites with uncertain parameters call for a probabilistic approach, which is time-consuming and computationally very expensive, so there is a need for the advanced method, which can quantify the uncertainty. The fuzzy based FEM, which is computationally fast and more reliable for the analysis of laminated composites plates with uncertain parameters.

The pertinent literature and state of the art of the problems available in the published literature in this area or related fields are reviewed in the next chapter.

Chapter -2

LITERATURE REVIEW

2.1 Introduction

The composite materials are used in the area where of the light weight of the structures play a vital roles such as in aerospace, automobile, marine and mechanical system. The basic parameter characterizing the response of laminated plate are many such as, fiber orientation, lamina thickness, voids, curing temperature and process. During manufacturing and fabrication, any variation in these parameters can leads to variation in the responses of the laminates. Due to the economic point of view, total quality control is practically not possible because a large number of parameter control are required. This lack of quality control results into randomness in the material and geometrical properties of the composite material.

The design and analysis of composite structures required that over the lifetime, it is performance is guaranteed. The majority of the structures fail because of the unreliable design and analysis. In composite structure, design process comprises of various parameters like material properties, boundary conditions, geometry and external loads are considered as deterministic in the nature. However, in reality they are not deterministic in nature rather imprecise, incomplete and linguistic (Chen and Rao, 1997), which ultimately cause an error in the response of the structure. The structural designer must provide an efficient design and innovative method to overcome this uncertainty (Muhanna and Mullen ,1999), where uncertainty in the system parameters must included while analyzing the system.

The available method for composite structural analysis i.e. FEM, in this method physical quantitates, are described using floating point number and response generated is non-fuzzy and crisp in nature. This idealized finite element model does not represent the uncertainty involved in the real physical model of the problem. Thus, to evolve a reliable and refined model, it is mandatory to take into consideration the uncertainties present in the real life structure. Therefore, there is an immediate need for an uncertainty centered model for the design and analysis of composite structures.

Plenty of the literature is available on design parameter variation such as geometry, load, material and boundary condition by using statistical and probabilistic method such and SFEM, MCS and perturbation approach. Moreover, in recent year's researchers have focused on used of fuzzy set theory for the analysis of engineering system with randomness in system parameter. A limited number of studies have reported on using the fuzzy concept in the analysis of composite structure with uncertainty in system parameter. In the following section, a comprehensive review of available literature is discussed. The presented literature review carried out for static and dynamic analysis of composite structures by using the fuzzy approach for including uncertainty in material properties. Along this, brief review for the application of the fuzzy approach for the various structural problem has been included for the sake of understanding of the robustness and applicability of the fuzzy method.

2.2 **Statistical and Probabilistic Methods**

Uncertainty in parameter mostly deals with statistical or stochastic uncertainty and is expressed by a probability density function and distribution assessment, that requires enough information of parameters and the variables that are uncertain, which are not known. Various researcher have studied the uncertainty in design parameter by using statistical and probabilistic method, some of them are discussed as under:

2.2.1 Static Analysis of Composite Structures

For the composite plates, first static analysis (Vanmarcke and Grigoriu, 1983) presented SFEM for solving engineering mechanics problems with random variation in physical properties. Salim, Yadav, and Iyengar (1993) used MCS to investigate the effect of material randomness in the response of the laminated composite plate. Van Vinckenroy and de Wilde (1995) developed a procedure to couple finite element modeling with MCS to deal with the uncertainty involves in composite structures. Lal and Singh (2011) presented study to obtain the flexural response of laminated composite plates with material property randomness in thermo-mechanical load using FEM-perturbation techniques. Dash and Singh (2015) investigated effects of material randomness on the transverse bending of laminated composite plates by using SFEM.

2.2.2 Dynamic Analysis of Composite Structures

For dynamic analysis of the composite structure Nakagiri, H., and Tani (1987) formulated methodology of SFEM for uncertain eigenvalue problem and investigated the vibration response of graphite/epoxy plate with simply supported boundary condition. Singh, Yadav, and Iyengar (2001) investigated the vibration response of the laminated composite plate with uncertainty in material properties using first order perturbation theory (FOPT) and validated the obtained results with MCS. Singh, Yadav, and Iyengar (2002) investigated influence of material properties variation in the natural frequency of laminated composite plate using C_0 FEM-perturbation method. Onkar and Yadav (2005) studied the forced nonlinear vibration response and sensitivity for lamina thickness and aspect ratio of the composite plate with material uncertainty. Shaker et al. (2008) performed reliability analysis of the free vibration of the composite plate with material uncertainty using first order reliability method and scored order reliability method. A review on the development of SFEM can be found in the article by (Singh and Neeraj, 2013). Review on the uncertainty based multidisciplinary design optimization with a focus on opportunities and challenges in the aerospace vehicles presented by (Yao et al., 2011).

Statistical or stochastic method for uncertainty quantification required number of samples to calculate the probable response of the structure are more costlier and time-consuming and increased the computational cost. Study present by (Wood, Antonsson, and Beck, 1990) suggested that probabilistic calculation is complex and fuzzy calculus and well suited for representing imprecision in the system parameter. This suggestion attracted the attention of researchers and new domain was introduced in the quantification of uncertainty by using fuzzy set theory, introduced by (Zadeh, 1965).

2.3 Fuzzy Finite Element Method

In FFEM, structural including with uncertainty is constructed by combining FEM in conjunction with fuzzy sets theory**.** Guang-yuan and Jin-ping (1990) presented mathematical theory of random vibration with fuzzy parameter and applied it to seismic structures. Dhingra, Rao, and Kumar (1992) presented fuzzy math programming to the multiple objective design of mechanical and structural system. Then onward many researcher started using fuzzy concept for the analysis of static and dynamic system with uncertainty in system parameter; few of them are discussed here:

2.3.1 Static Analysis of Structures

In the early 1990, (Rao and P. 1995) formulated FFEM for the analysis of the imprecisely defined system described by inaccurate/incomplete information of geometry, boundary condition, load and material properties. Elishakoff (1998) advocated that hybrid optimization approach is better for a realistic structure with uncertainty. Rao and Chen (1998) presented methodology and procedure for enhancing the speed and accuracy of the fuzzy solution. Muhanna and Mullen (1999) presented fuzzy approach for uncertainty treatment in continuum mechanics, with numerical example considering applied load, material properties, and geometric uncertainty. Lallemand et al. (1999) proposed new framework to predict structures effective properties based on fuzzy sets and extended fuzzy set theory to the dynamic analysis of the system with material uncertainty. Akpan et al. (2001) formulated and developed practical FFEA procedure to investigate the static response of engineering structure with fuzziness in system parameter by combining fuzzy analysis, response surface, and finite element modeling. Rama Rao and Ramesh Reddy (2007) used FFEM to get fuzzy static response of a cable-stayed bridge with uncertainties. Franck Massa, Tison, and Lallemand (2006) formulated efficient method for improving the prediction of the numerical model with vague information for static analysis integrated in the design stage. Balu and Rao (2012) used FFEM to investigate static and dynamic responses of structures. Recently, uncertain static responses of imprecisely defined structures are studied by using FFEM by (Behera and S., 2013).

2.3.2 Dynamic Analysis of Structures

Chen and Rao (1997) proposed FFEM for the vibration analysis of the vaguely defined system supported with numerical examples. Cherki et al. (2000) investigated mechanical systems with fuzzy behaviour considering uncertain boundary condition. D. Moens and Vandepitte (2002) presented FFEM for dynamic design validation by considering four case studied to advocate the applicability of interval finite element and FFEM for dynamic analysis of structure involving different types of the uncertainties. Envelope frequency response functions calculated by (D. Moens and Vandepitte, 2005) using interval finite element approach. Transformation method were applied by (Giannini and Hanss, 2008) to characterize the dynamic behavior of structure with fuzzy parameter with application to the beam problem. F. Massa, Tison, and Lallemand (2009) presented comparison of fuzzy based numerical and experimental data to forecast the change in the model behavior, for two plate structure. Balu and Rao (2012) proposed methodology for calculating structures response with fuzzy parameter based on high dimensional model representation. Adhikari and Khodaparast (2014) proposed spectral method for fuzzy uncertainty propagation for increasing computational efficiency. Xia and Friswell (2014) presented modified interval perturbation method applicable for eigenvalue problem in structural dynamics.

A review of recent development on non-probabilistic method for static and dynamic analysis can be found in the article by (MacE, Vandepitte, and Lardeur, 2005), (D Moens and Vandepitte, 2006), (David Moens and Hanss, 2011) and (Simoen, De Roeck, and Lombaert, 2015).

2.4 Fuzzy Finite Element Method in Composite Structures

Noor, Starnes, and Peters (2000) presented step by step procedure and two-phase approach for forecasting the variation in the response of composite structures with geometric and material parameter uncertainty of cylindrical panel by using fuzzy set. Liu and Rao (2003) presented a fuzzy approach for modeling the randomness in the mechanics of the composite materials. The fuzzy operation used for modification of composite mechanics, fuzzy properties of the lamina and derived the law of mechanics for thin orthotropic lamina. Liu and Rao (2005) developed fuzzy beam element using fuzzy matrix operation, deterministic finite element theory, and fuzzy computation. Then FFEM described for the analysis of static and eigenvalue of the laminated composite beam with uncertainty in material properties based on deterministic finite element theory and fuzzy arithmetic. Pawar, Jung, and Ronge (2012) used fuzzy approach for deriving the solution for thin composite beam structures, is utilized in the analysis of thin wall composite beam having material uncertainty. The presented approach demonstrated by obtaining membership function of cross-sectional stiffness material properties of the beam using membership function of material properties.

Based on the presented literature it is clear that the fuzzy finite element study is limited to laminated beam using classical plate theory only. The vibration, bending and buckling behavior of the laminated structure are computed using uncertain material properties based on probabilistic method, and no study has been reported yet for the analysis of the laminated composite plate based on framework of higher order shear deformation theory (HSDT).

Chapter- 3

FUZZY SET THEORY

In this chapter definitions of a fuzzy set, interval, triangular fuzzy numbers, the notations and fuzzy/interval arithmetic are presented in relevance to the present study. Several books and research articles related to fuzzy set theory are available, which express the scope and various traits of interval and fuzzy set theory such as (Zadeh, 1965) (Liu and Rao, 2005) and (Moore, R, and M, 2009).

3.1 Definition of Fuzzy Set

A fuzzy set $\tilde{U}$ is defined by set of ordered pairs such that

$$\tilde{U} = \{(u, \mu_{\tilde{U}}(u)) \mid u \in \mathbb{R}, \mu_{\tilde{U}}(u) \in [0, 1]\}, \tag{3.1}$$

where, $\mu_{\tilde{U}}(u)$ is named, as the degree of compatibility or membership function.

3.2 Definition of Interval

An interval u is denoted by $[\underline{u}\ \overline{u}]$ in real numbers set $\mathbb{R}$ given by

$$\tilde{u} = [\underline{u}\ \overline{u}] = \{u \in \mathbb{R} : \underline{u} \leq u \leq \overline{u}\}. \tag{3.2}$$

Although there exist various types of intervals such as open and half open intervals. Here $\underline{u}$ left bound and $\overline{u}$ right bound of the $\tilde{u}$ interval.

3.3 Definition of Fuzzy Arithmetic

Let's take two arbitrary intervals $\tilde{u} = [\underline{u}\ \overline{u}]$ and $\tilde{v} = [\underline{v}\ \overline{v}]$

Then fuzzy addition (+), subtraction (-), multiplication (.) and division (/) are defined as follows: $\tilde{u}(+)\tilde{v} = [\underline{u} + \underline{v}, \bar{u} + \bar{v}]$,

(3.3 $\tilde{u}(-)\tilde{v} = [\underline{u} - \bar{v},\ \bar{u} - \underline{v}]$, (3.4)

$$\tilde{u}(.)\tilde{v} = \left[\min T, \max T\right], \tag{3.5}$$

where $T = \{\underline{u} \times \underline{v}, \underline{u} \times \bar{v}, \bar{u} \times \underline{v}, \bar{u} \times \bar{v}\}$,

$$\tilde{u}(/)\tilde{v} = [\underline{u}, \bar{v}].[\frac{1}{\bar{v}}, \frac{1}{\underline{v}}] \text{ if } 0 \notin \tilde{y}. \tag{3.6}$$

Now if L is a real number and $L \in \mathbb{R}$

Multiplication of real number with fuzzy number $\tilde{x} = [\underline{x}, \bar{x}]$, given by

$$L\tilde{x} = \begin{cases} [L\bar{x}, L\underline{x}], L < 0, \\ [L\underline{x}, L\bar{x}], L \geq 0. \end{cases} \tag{3.7}$$

3.4 Definition of Fuzzy Number

A fuzzy number $\tilde{U}$ is a convex normalized fuzzy set $\tilde{U}$ of the real line $\mathbb{R}$ such that

$$\{\mu_{\tilde{U}}(u) : \mathbb{R} \to [0,\ 1],\ \forall\ u \in \mathbb{R}\} \tag{3.8}$$

where, $\mu_{\tilde{U}}$ is so-called the membership function and it is piecewise continuous. There exists a variety of fuzzy numbers. However, in this study only the triangular fuzzy numbers used. So, we define only triangular fuzzy numbers below.

3.5 Definition of Triangular Fuzzy Number

Let us consider an arbitrary triangular fuzzy number $\tilde{U} = (a, b, c)$ as shown in Fig. 3.1. Then the membership function $\mu_{\tilde{U}}$ of the fuzzy number $\tilde{U}$ will be:

$$\mu_{\tilde{U}}(u) = \begin{cases} 0, & u \leq a \\ \dfrac{u-a}{b-a}, & a \leq u \leq b \\ \dfrac{c-u}{c-b}, & b \leq u \leq c \\ 0, & u \geq c. \end{cases} \tag{3.9}$$

And the triangular fuzzy number $\tilde{U} = (a, b, c)$ can be represented in interval functions using $\alpha -$cut method.

$$[\underline{u}(\alpha), \overline{u}(\alpha)] = [(b-a)\alpha + a, \ -(c-b)\alpha + c]$$

where, $\alpha \in [0,1]$.

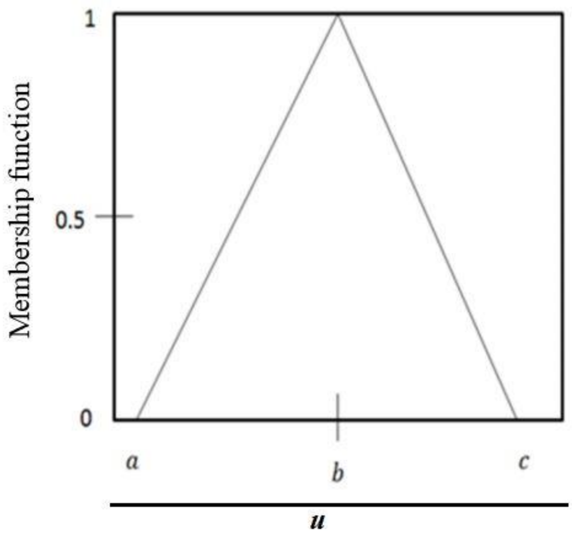


Figure 3.1 Triangular fuzzy number

Chapter - 4

GENERAL MATHEMATICAL FORMULATION

4.1 Fuzzy Kinematics Model

This chapter deal with the theory and fuzzy finite element formulation for free vibration and bending analysis of laminated composite plate. The governing equation for fuzzy based FEM for static and dynamic analysis of laminated composite plate are derived in the framework of HSDT.

A laminated composite is constructed by stacking number of laminas in the direction of thickness Z and the geometric mid plane z= (0) of laminate is in the XY plane.

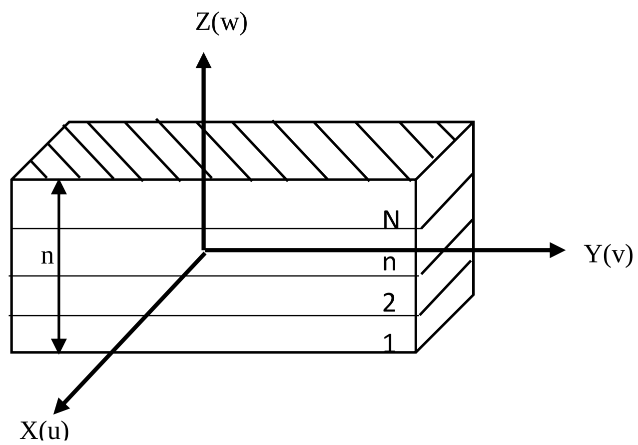


Figure 4.1 Laminate geometry

The total number of laminas is *N* and *h* and total thickness, as shown in Fig.4.1 It is necessary to mention that with introduction to fuzzy arithmetic operation (Liu and Rao, 2005) which include fuzzy addition (+), subtraction (-), multiplication (.) and division (/), derivation and integration for the laminated composite plate; used for deriving the displacement field based on the HSDT (Reddy, 2003), which is derived for deterministic structure. It can be extened to the estimation of uncertaniteis in the laminated composite plate without loss of generality (Pawar, Jung, and Ronge, 2012). Every term in derivation having subscript α indicate two α-cut values namely lower and upper bound. The plate coordinate system represented using Cartesian coordinate system (x,y and z). the displacement field of any point in the composite plate at a distance z from the mid plane is assumed in the form of:

$$u_\alpha(x,y,z,t) = u_{0\alpha}(x,y)(+)z(.)\theta_{x\alpha}(x,y)(+)z^2(.)\phi_{x\alpha}(x,y)(+)z^3(.)\lambda_{x\alpha}(x,y)$$
$$v_\alpha(x,y,z,t) = v_{0\alpha}(x,y)(+)z(.)\theta_{y\alpha}(x,y)(+)z^2(.)\phi_{y\alpha}(x,y)(+)z^3(.)\lambda_{y\alpha}(x,y)$$
$$w_\alpha(x,y,z,t) = w_{0\alpha}(x,y) \tag{4.1.1}$$

where, fuzzy displacements of a point along the (x, y, and z) coordinates represented by u_α, v_α and w_α and, t is the time. Fuzzy displacements of a point at the mid plane of the plate denoted by $u_{0\alpha}$, $v_{0\alpha}$ and $w_{0\alpha}$. Fuzzy rotations of normal to the mid plane about the y and x-axes indicated by $\theta_{x\alpha}$ *and* θ_{y_α} respectively. Higher-order terms in the Taylor series expansion defined in the mid plane of plate represent by functions $\phi_{x\alpha}, \phi_{y\alpha}, \lambda_{x\alpha}, \lambda_{y\alpha}$.

The fuzzy strain–displacement relation for the fuzzy-based laminated plate derived from the standard strain-displacement relation as in (Cook et al., 2007):

$$\{\varepsilon_\alpha\} = \begin{Bmatrix} \varepsilon_{xx\alpha} \\ \varepsilon_{yy\alpha} \\ \varepsilon_{zz_\alpha} \\ \varepsilon_{yz\alpha} \\ \varepsilon_{xz_\alpha} \\ \varepsilon_{xy_\alpha} \end{Bmatrix} = \begin{Bmatrix} \left(\frac{\partial u_\alpha}{\partial x}\right) \\ \left(\frac{\partial v_\alpha}{\partial y}\right) \\ \left(\frac{\partial w_\alpha}{\partial z}\right) \\ \left(\frac{\partial v_\alpha}{\partial z}(+)\frac{\partial w_\alpha}{\partial y}\right) \\ \left(\frac{\partial u_\alpha}{\partial z}(+)\frac{\partial w_\alpha}{\partial x}\right) \\ \left(\frac{\partial u_\alpha}{\partial y}(+)\frac{\partial v_\alpha}{\partial x}\right) \end{Bmatrix} \tag{4.1.2}$$

Substituting Eq. (4.1.1) in Eq. (4.1.2). The total fuzzy strain can given in terms of displacement and conceded as:

$$\{\varepsilon_\alpha\} = \begin{Bmatrix} \varepsilon_{1\alpha}{}^{l_0} \\ \varepsilon_{2\alpha}{}^{l_0} \\ \varepsilon_{3\alpha}{}^{l_0} \\ \varepsilon_{4\alpha}{}^{l_0} \\ \varepsilon_{5\alpha}{}^{l_0} \\ \varepsilon_{6\alpha}{}^{l_0} \end{Bmatrix} (+)\, z(.) \begin{Bmatrix} k_{1\alpha}{}^{l_1} \\ k_{2\alpha}{}^{l_1} \\ 0 \\ k_{4\alpha}{}^{l_1} \\ k_{5\alpha}{}^{l_1} \\ k_{6\alpha}{}^{l_1} \end{Bmatrix} (+)z^2(.) \begin{Bmatrix} k_{1\alpha}{}^{l_2} \\ k_{2\alpha}{}^{l_2} \\ 0 \\ k_{4\alpha}{}^{l_2} \\ k_{5\alpha}{}^{l_2} \\ k_{6\alpha}{}^{l_2} \end{Bmatrix} (+)z^3(.) \begin{Bmatrix} k_{1\alpha}{}^{l_3} \\ k_{2\alpha}{}^{l_3} \\ 0 \\ k_{4\alpha}{}^{l_3} \\ k_{5\alpha}{}^{l_3} \\ k_{6\alpha}{}^{l_3} \end{Bmatrix} \tag{4.1.3}$$

Now fuzzy strain–displacement relation can be rearranged in matrix form:

$$\{\varepsilon_\alpha\} = [T](.)\{\overline{\varepsilon_\alpha}\} \tag{4.1.4}$$

Where,

$$\left\{\overline{\varepsilon_\alpha}\right\} = \left\{\varepsilon_{1\alpha}{}^{l_0}\varepsilon_{2\alpha}{}^{l_0}\varepsilon_{3\alpha}{}^{l_0}\varepsilon_{4\alpha}{}^{l_0}\varepsilon_{5\alpha}{}^{l_0}\varepsilon_{6\alpha}{}^{l_0}k_{1\alpha}{}^{l_1}k_{2\alpha}{}^{l_1}k_{4\alpha}{}^{l_1}k_{5\alpha}{}^{l_1}k_{6\alpha}{}^{l_1}k_{1\alpha}{}^{l_2}k_{2\alpha}{}^{l_2}k_{4\alpha}{}^{l_2}k_{5\alpha}{}^{l_2}k_{6\alpha}{}^{l_2}k_{1\alpha}{}^{l_3}k_{2\alpha}{}^{l_3}k_{4\alpha}{}^{l_3}k_{5\alpha}{}^{l_3}k_{6\alpha}{}^{l_3}\right\}^T$$

and $[T]$ is the function of thickness coordinate matrices. $\left\{\overline{\varepsilon_\alpha}\right\}$ Containing superscripts 'l_0', 'l_1', 'l_2-l_3' are the fuzzy membrane, fuzzy curvature and fuzzy higher order strain terms, respectively.

The general constitutive relation for any arbitrary k^{th} orthotropic lamina in the laminate with fuzzy α-cut is given by:

$$\begin{Bmatrix} \sigma_{x\alpha} \\ \sigma_{y\alpha} \\ \sigma_{z\alpha} \\ \tau_{yz\alpha} \\ \tau_{xz\alpha} \\ \tau_{xy\alpha} \end{Bmatrix}^k = \begin{Bmatrix} \sigma_{1\alpha} \\ \sigma_{2\alpha} \\ \sigma_{3\alpha} \\ \sigma_{4\alpha} \\ \sigma_{5\alpha} \\ \sigma_{6\alpha} \end{Bmatrix}^k = \begin{bmatrix} Q_{11\alpha} & Q_{12\alpha} & Q_{13\alpha} & 0 & 0 & Q_{16\alpha} \\ Q_{12\alpha} & Q_{22\alpha} & Q_{23\alpha} & 0 & 0 & Q_{26\alpha} \\ Q_{13\alpha} & Q_{23\alpha} & Q_{33\alpha} & 0 & 0 & Q_{36\alpha} \\ 0 & 0 & 0 & Q_{44\alpha} & Q_{45\alpha} & 0 \\ 0 & 0 & 0 & Q_{45\alpha} & Q_{55\alpha} & 0 \\ Q_{16\alpha} & Q_{26\alpha} & Q_{36\alpha} & 0 & 0 & Q_{66\alpha} \end{bmatrix} (.) \begin{Bmatrix} \varepsilon_{1\alpha} \\ \varepsilon_{2\alpha} \\ \varepsilon_{3\alpha} \\ \varepsilon_{4\alpha} \\ \varepsilon_{5\alpha} \\ \varepsilon_{6\alpha} \end{Bmatrix}^k \tag{4.1.5}$$

where, $\left[Q_{ij\alpha}\right]^k$ are reduced fuzzy material constant for any orthotropic k_{th} layer. The uncertainty material properties can be introduced in the constitutive relation through, $\left[Q_{ij\alpha}\right]^k$ stiffness coefficient of k^{th} layer in the composite (Liu and Rao, 2003).

The total strain energy of the laminated composite plate can be obtained by substituting the values of stresses and strains from the Eq. (4.1.4) and Eq. (4.1.5) presented as follows:

$$U = \frac{1}{2}\int_A \left\{\varepsilon_\alpha\right\}_i^T (.) \left\{\sigma_{i\alpha}\right\} dA = \frac{1}{2}\int_A \left\{\varepsilon_\alpha\right\}_i^T (.) \left[\overline{Q_\alpha}\right] (.) \left\{\varepsilon_\alpha\right\}_i dA$$

$$U = \frac{1}{2}\int_A \left(\left\{\varepsilon_\alpha\right\}_i^T (.) \left[D_\alpha\right] (.) \left\{\varepsilon_\alpha\right\}_i \right) dA \tag{4.1.6}$$

where, $[D_\alpha]$ is element laminate stiffness matrix

then, th kinetic energy of the laminates expressed as:

$$T = \frac{1}{2}\int\int \left[\sum_{k=1}^{N} \int_{h_k}^{h_{k+1}} \rho_{k\alpha} (.) \left\{\dot{\delta}_\alpha\right\} (.) \left\{\dot{\delta}_\alpha\right\}^T dz \right] dxdy \tag{4.1.7}$$

where, $\rho_{k\alpha}$, h_k, h_{k+1}b and N is mass density, z coordinates of laminates corresponding to top and bottom surface of the k^{th} layer and number of lamina respectively.

The kinetic energy for laminated composite plate with N of lamina will be:

$$V = \frac{1}{2}\int_A \left\{\dot{\delta}_\alpha\right\}^T (.)\left[m_\alpha\right](.)\left\{\dot{\delta}_\alpha\right\} dA \tag{4.1.8}$$

where, $\left[m_\alpha\right] = \sum_{k=1}^{N}\int_{Z_{k-1}}^{Z_k} \left[f\right]^T (.)\rho_\alpha{}^k (.)\left[f\right] dz$ is elemental inertia matrix.

The governing equation of laminated composite plate is obtained using Hamilton's principle (Reddy, 2003) can be expressed as:

$$\delta\int_{t_1}^{t_2} \left[T(-)(U(+)V)\right] dt = 0 \tag{4.1.9}$$

Now, substituting, the value of U, T, and V from Eq. (4.1.6), (4.1.7), (4.1.8) and performing the fuzzy arithmetic operation (Liu and Rao, 2005), Eq. (4.1.9) can be expressed in the matrix form as:

$$\left[M_\alpha\right](.)\left\{\ddot{\delta}_\alpha\right\}(+)\left(\left[K_\alpha\right]\right)(.)\left\{\delta_\alpha\right\} = \left\{F_\alpha\right\} \tag{4.1.10}$$

where, $[M_\alpha]$, $[K_\alpha]$, $\{F_\alpha\}$ and $\left\{\delta_\alpha\right\}$ are the global fuzzy mass matrix, global fuzzy stiffness matrix, global fuzzy force vector and global displacement vector respectively. Global fuzzy mass matrix $[M_\alpha]$ and global fuzzy stiffness matrix, $[K_\alpha]$ are functions of a structural parameter like material, geometry, and mass characteristics.

4.2 Fuzzy Finite Element Analysis

A nine noded isoparametric element employed for the static and dynamic analysis of laminated composite plate. The domain discretized into a set of finite elements and nine degree of freedom considered at every node. Over each of the elements, the fuzzy displacement vector $\left\{\delta_\alpha\right\}$ form by using the FEM:

$$\left\{\delta_\alpha\right\} = \left[N_i\right](.)\left\{\delta_\alpha\right\}_i, \left\{\bar{\varepsilon}_\alpha\right\}_i = \left[B\right]_i (.)\left\{\delta_\alpha\right\}_i \tag{4.2.1}$$

where, $[N_i]$, $[B]$ and $\{\delta_\alpha\}_i$ are interpolation functions for the i^{th} node, nodal interpolation function in the strain terms, the product form of the fuzzy differential operator and vector of unknown displacements for the i_{th} node respectively.

Now, substituting the interpolation functions of a nine noded isoparametric element from Eq. (4.2.1), the total strain energy will be given as follows:

$$U=\frac{1}{2}\int_A\left(\left\{\delta_\alpha\right\}_i^T(.)\left[B\right]_i^T(.)\left[D_\alpha\right](.)\left[B\right]_i(.)\left\{\delta_\alpha\right\}_i\right)dA \qquad (4.2.2)$$

In this study, material properties of laminated plates are random in nature and treated as fuzzy using triangular membership function. As a result displacement vector $\left\{\delta_\alpha\right\}$ also becomes fuzzy in nature.

For the natural frequency calculation of the laminated composite plate, Eq. (4.1.10) transformed as:

$$\left[K_\alpha\right](.)\left\{\delta_\alpha\right\}=\lambda_\alpha(.)\left[M_\alpha\right](.)\left\{\delta_\alpha\right\} \qquad (4.2.3)$$

Similarly, for the static response of the laminated composite plate under applied load, Eq. (4.1.10) written in the most general form as:

$$\left[K_\alpha\right](.)\left\{\delta_\alpha\right\}=\left\{F_\alpha\right\} \qquad (4.2.4)$$

where, λ_α is eigenvalue, equal to ω^2.

The global stiffness matrix [K_α] is having uncertainty and fuzziness at each α-cut. In summary, as a first step, uncertainty in the material properties of laminated composite plate introduced in the model by modeling uncertain parameter as a fuzzy number. There are many types of the membership function, but the triangular membership function used for the simple reason of the computational simplicity. Once uncertainty introduced in the model, the domain is fuzzified, then the element matrices obtain in the form of fuzzy matrix, whose elements are a fuzzy number. After evaluating element matrices, assembled them to get system matrices that result as fuzzy equations. Fuzzy equations are solved using an interval of confidence at finite α-cut (Rao and P., 1995), which reduced to a solving system of interval equations. In the model external loads and boundary condition applied as per the deterministic FEA procedure.

Chapter- 5

RESULTS AND DISCUSSIONS

5.1 Introduction

As discussed in the chapter one, the laminated composite plates used in a variety of engineering structures like space vehicle, automobile, sports utilities. A number of design parameter are involved during the manufacturing and assemblage of the composite plates. The complete quality control is economically not viable to maintain, so it becomes very important to investigate the effect of the random material properties on the static response and natural frequency of the composite plate.

This chapter, includes the convergence and validation of the laminated composite beam model, which was further extended to the convergence and validation of the proposed model for the laminated composite plate. Several numerical examples and comparisons with published result performed to demonstrate the applicability and accuracy of the proposed model. The effects of various design parameters such as the lamination schemes and the thickness ratios (a/h), on the free vibration and the bending behavior of the composite plate, have presented.

5.2 Convergence and Validation

The convergence and validation of the present developed code has been demonstrated by comparing the result with those available in the literature. The geometrical and material properties of the laminates are taken same as in (Liu and Rao, 2005). The convergence study is performed by varying the number of the elements used in the proposed model. Fig. 5.1 shows the convergence study of a cantilever composite $[+45^0/-45^0]_s$ beam of length 2m and width 0.024m. The displacements at the free end of the beam are computed for different mesh sizes. It is observed that 18 elements are sufficient to get the desired response.

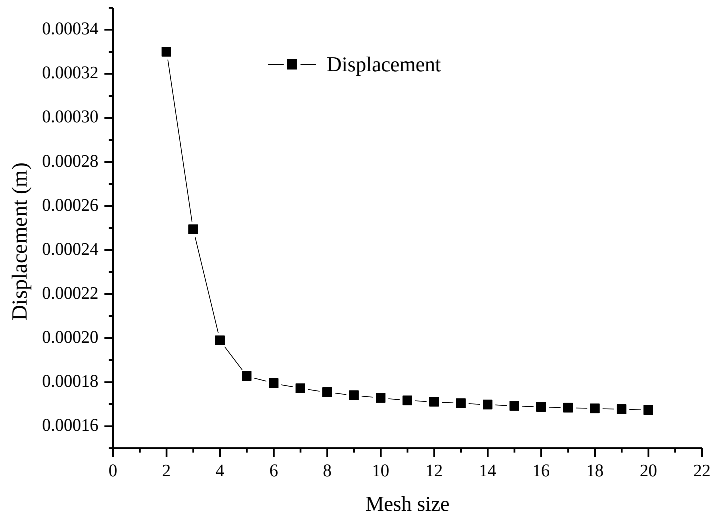


Figure 5.1 Convergence study of the displacement of laminated composite beam

The validation of the fuzzy finite element technique used in the present problem is carried out by comparing the present results with the previous published literature (Liu and Rao, 2005) as in Table 5.1.

Table 5.1 Comparison study of the displacement of the cantilever beam

α	**Present (18 element)**	**(Liu and Rao, 2005)**	**Difference (%)**
0	0.00016458	0.000145628	13.0
0.2	0.00016522	0.000145628	13.4
0.4	0.00016625	0.000145628	14.1
0.6	0.0001672	0.000145628	14.8
0.8	0.00016768	0.000150875	11.1
1	**0.00016797**	**0.000153799**	**9.2**
0.8	0.00016814	0.000155997	7.7
0.6	0.00016878	0.000161063	4.7
0.4	0.00016965	0.000161063	5.3
0.2	0.00017069	0.000161063	5.9
0	0.0001715	0.000161063	6.4

The material properties are modeled through triangular membership function. From the table, it is clearly visible that the results obtained through fuzzy finite element formulation is having good agreement with the published results. It is also noticed that for α=1 i.e., deterministic value, nominal difference between the present results and the published results is observed. This is because of the fact that the present model is based on the HSDT whereas the classical laminate beam theory is used in (Liu and Rao, 2005).

5.3 Vibration Analysis of The Laminated Composite Plate

In this section, the natural frequency of the laminated composite plate with uncertain material properties is obtained. Firstly the developed code is validated by comparing the present results with those available in the literature. Secondly, a detailed parametric study has been performed for the laminated composite plate, to check the efficacy of the present numerical model with the discussion of result.

The problem of square laminated composite plate undergoing free vibrations is considered. The framework of HSDT is used to calculate the natural frequency of the composite plate. The randomness in material properties is introduced using the fuzzy membership function as described in Chapter 3. The geometrical and material properties are taken from (Singh, Yadav, and Iyengar, 2001) for the natural frequency calculation. Geometry and material properties are as follows:

Material-I: E_{11} =25 Gpa, E_{22} =1 Gpa, G_{12}= G_{13}=0.5E_{22}, G_{23}=0.2E_2, v_{12}= v_{13}= v_{23}=0.25

Material-II: E_{11} =40 Gpa, E_{22} =1 Gpa, G_{12}= G_{13}=0.6E_{22}, G_{23}=0.5E_2, v_{12}= v_{13}= v_{23}=0.25

Uncertainty in the material properties like the longitudinal modulus, transverse modulus, shear modulus etc. are considered to be random variables and taken in the range of [±20%] about their deterministic values. The triangular membership function used for modeling the uncertain material properties and presented in Fig. 5.2 and Fig. 5.3.

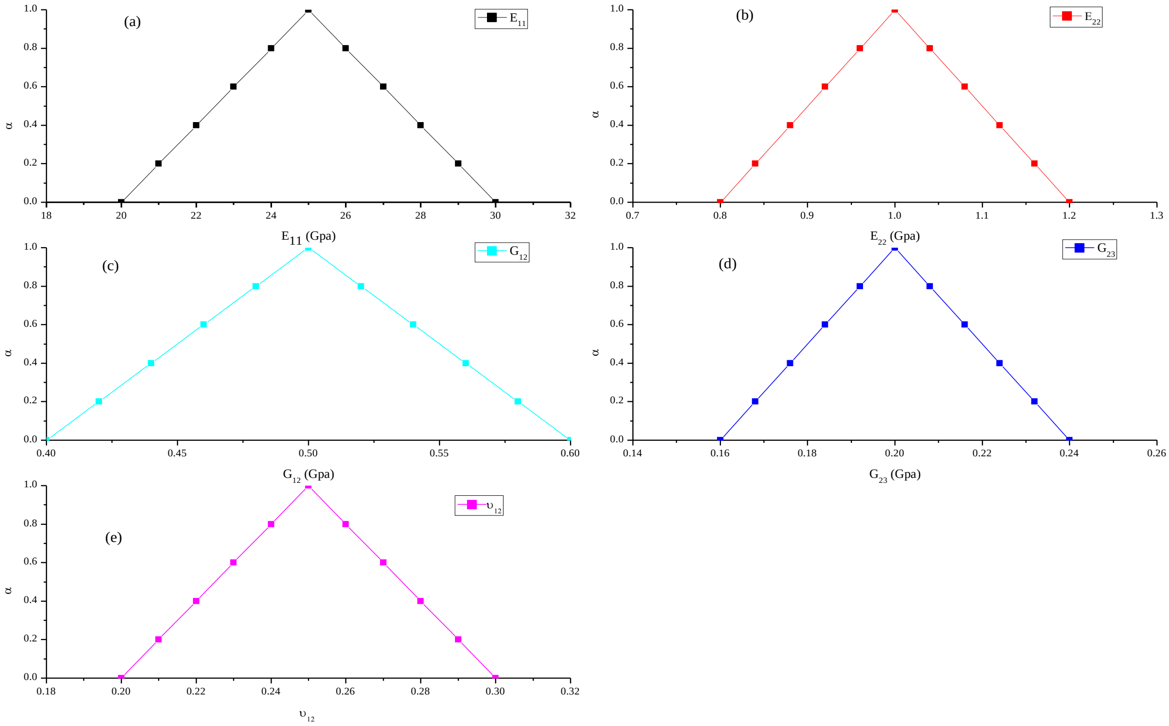


Figure 5.2 Membership function of the material properties of Material – I

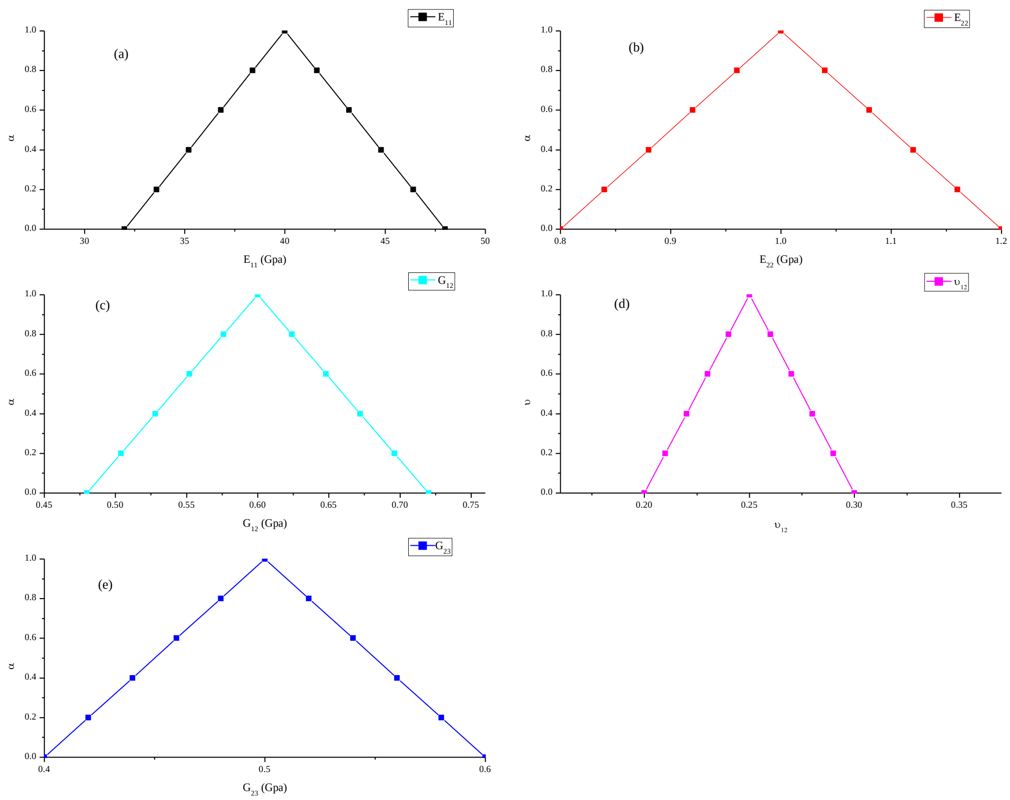


Figure 5.3 Membership function of the material properties of Material – II

The non dimensionalisation formula used for analysis is as follows:

$$\left(\varpi\right) = \left(\omega a^2 \sqrt{\frac{\rho}{E_{22}}}\right) / h$$

5.3.1 Convergence and validation study for the nondimensional natural frequency of the laminated composite plate

The convergence behavior of the present developed mathematical model has checked for the nondimensional natural frequencies and presented in Fig. 5.4.

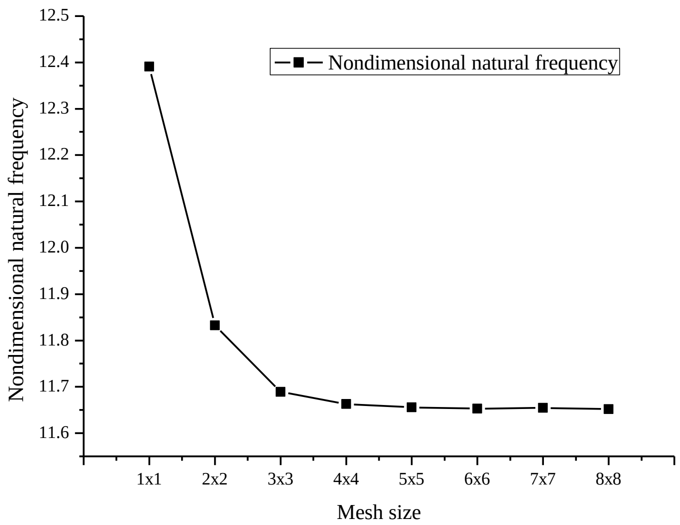


Figure 5.4 Convergence study of nondimensional natural frequency of laminated composite plate

Based on the convergence study it is assumed that a (5×5) mesh size is appropriate to calculate the responses of the laminated composite plate, hence a (5×5) mesh size used for computing the results. Validation of the fuzzy finite element technique used for the present problem was done by comparing the present result for α=1(deterministic value) with the result obtained from the analysis of laminated composite plate in ANSYS parametric design language (APDL) and of the crisp value of nondimensional natural frequency from (Singh, Yadav, and Iyengar, 2001) are presented in Table 5.2. It can be observed from the table that the crisp value of the nondimensional natural frequency of the laminated composite plate is very close to the value obtained by FFEA with α=1. The nondimensional natural frequency of the laminated composite plate with the fuzzified material properties, calculated for different amplitude ratios. Fig. 5.5-5.8 represent the nondimensional natural frequency distribution for lamination scheme [$0^0/90^0$] and [$0^0/90^0/90^0/0^0$] with the material I for a/h=10 and a/h=100 respectively. Fig 5.9-5.12 represent the nondimensional natural frequency distribution for lamination scheme [$0^0/90^0$] and [$0^0/90^0/90^0/0^0$] with material II for a/h=10 and a/h=100 respectively.

Table 5.2 Comparison study of nondimensional natural frequency of simply supported laminated composite plate

Material- I						
	a/h=10			**a/h=100**		
	FFEM (α=1)	**(APDL)**	(Singh, Yadav, and Iyengar, 2001)	**FFEM (α=1)**	**APDL**	(Singh, Yadav, and Iyengar, 2001)
[$0^0/90^0$]	8.89	8.89	8.98	9.72	9.68	10.47
[$0^0/90^0/90^0/0^0$]	11.65	11.54	11.77	15.16	15.16	15.17
Material - II						
	a/h=10			**a/h=100**		
[$0^0/90^0$]	10.38	10.24	10.56	11.37	11.29	11.90
[$0^0/90^0/90^0/0^0$]	15.03	15.02	15.10	18.89	18.83	19.13

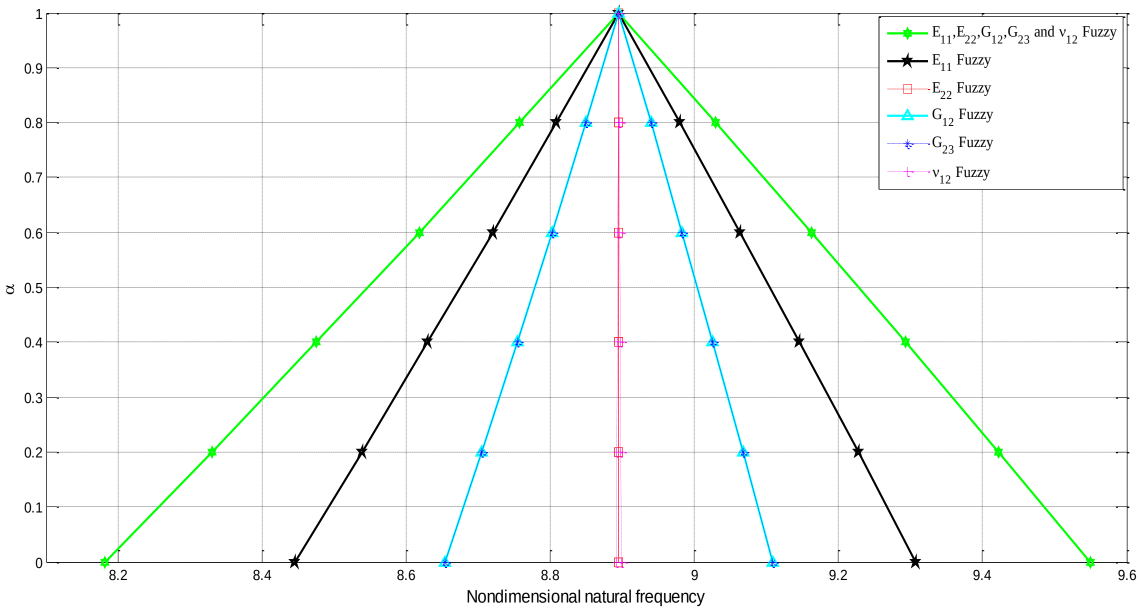


Figure 5.5 Nondimensional natural frequency distribution for a/h=10, [$0^0/90^0$], Material-I

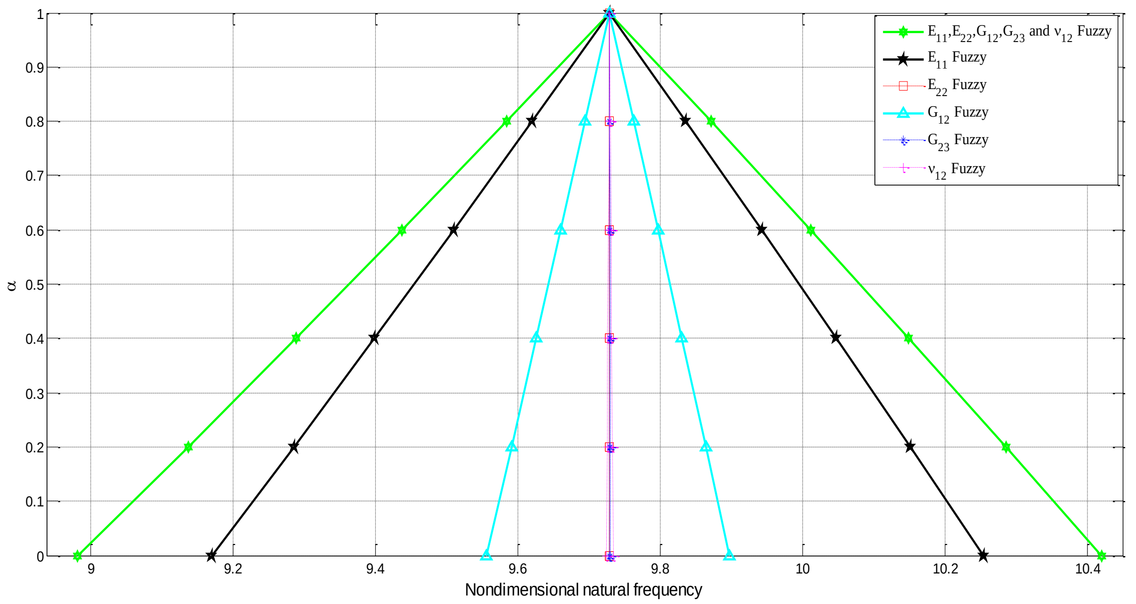


Figure 5.6 Nondimensional natural frequency distribution for a/h=100, [$0^0/90^0$], Material-I

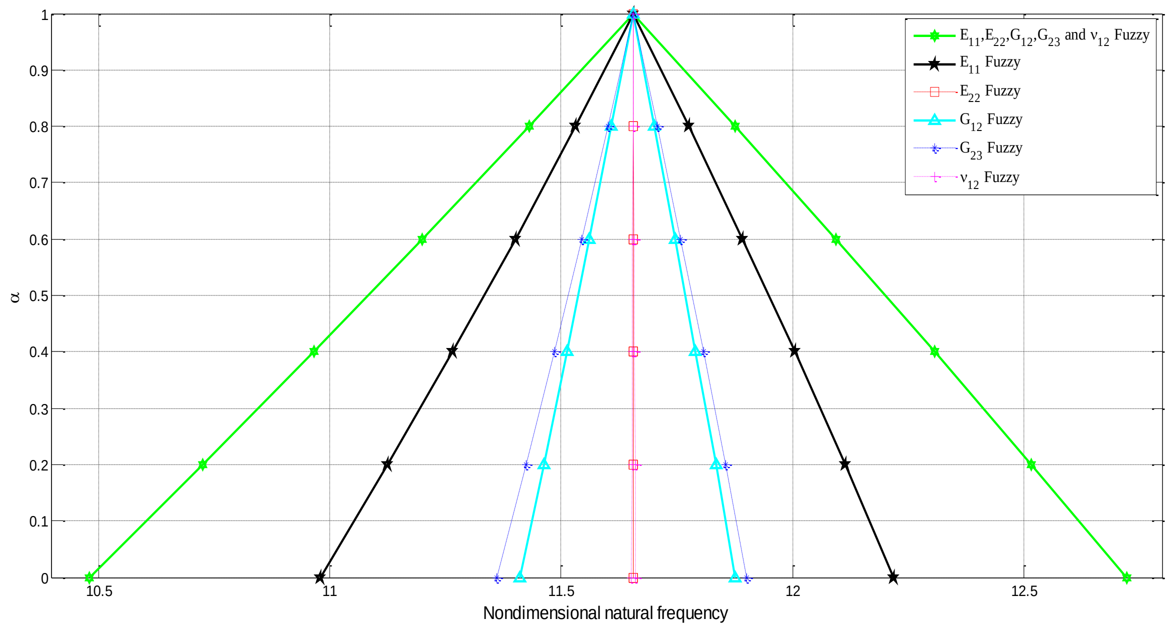


Figure 5.7 Nondimensional natural frequency distribution for a/h=10, [$0^0/90^0/90^0/0^0$], Material-I

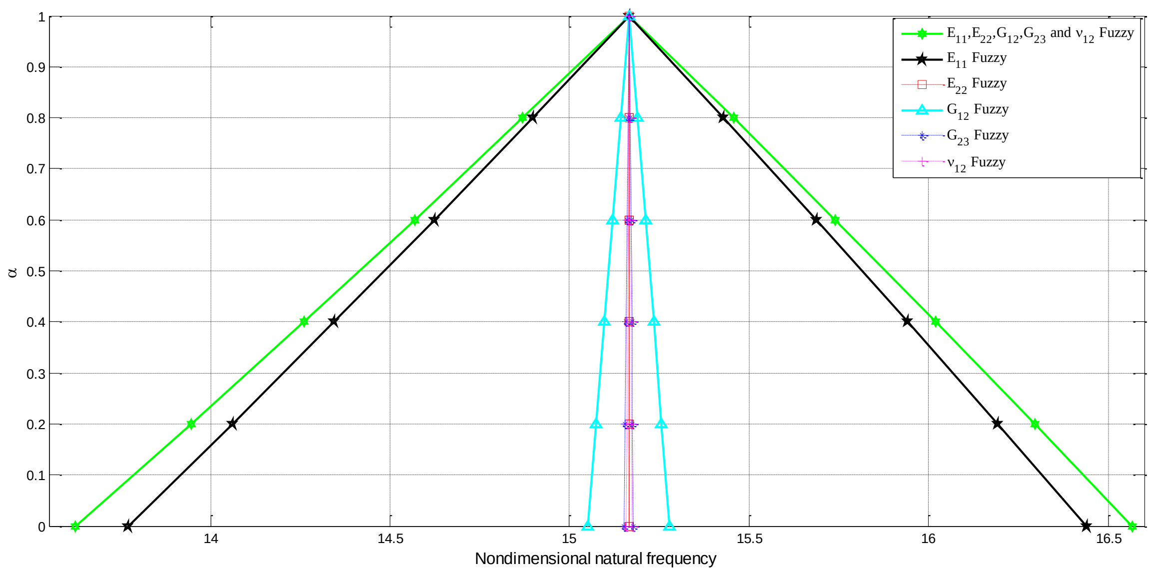


Figure 5.8 Nondimensional natural frequency distribution for a/h=100, [$0^0/90^0/90^0/0^0$], Material-I

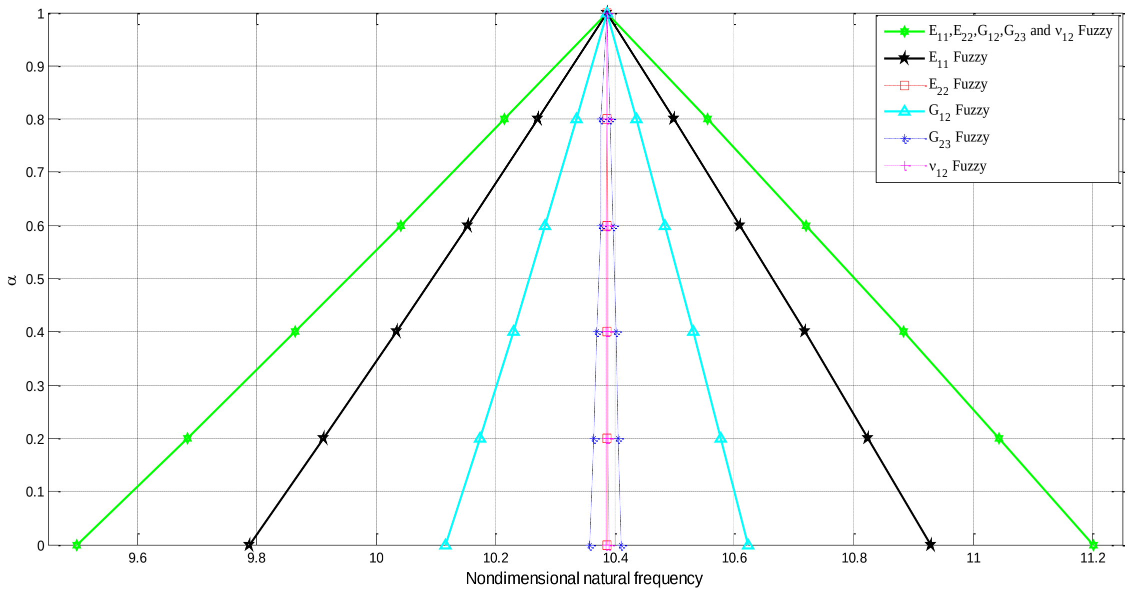


Figure 5.9 Nondimensional natural frequency distribution for a/h=10, [0^0/90^0], Material-II

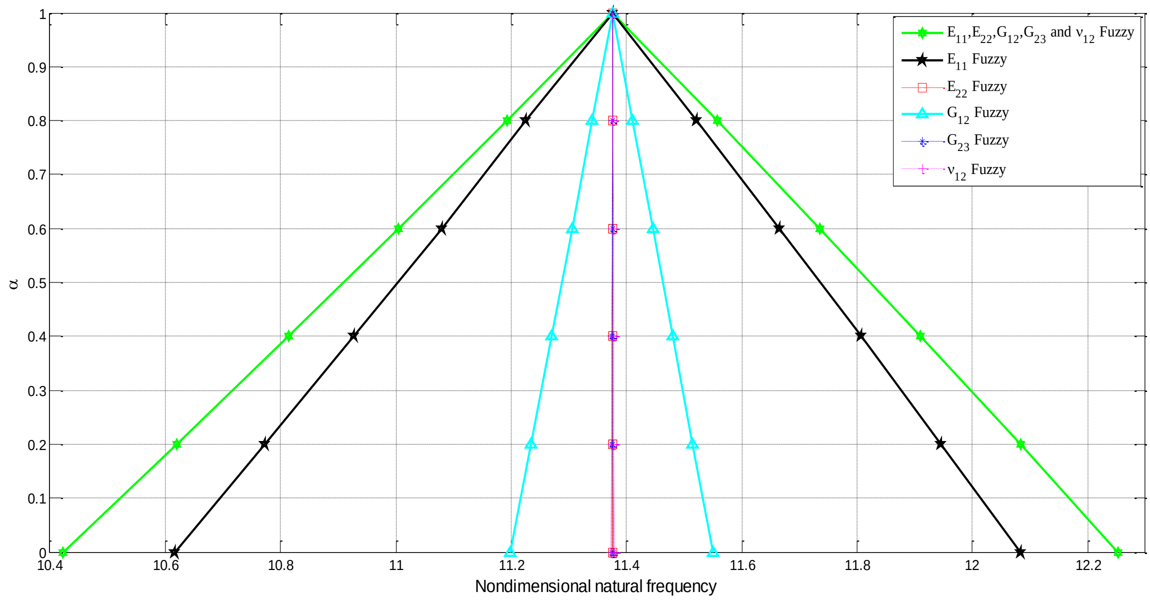


Figure 5.10 Nondimensional natural frequency distribution for a/h=100, [0^0/90^0], Material-II

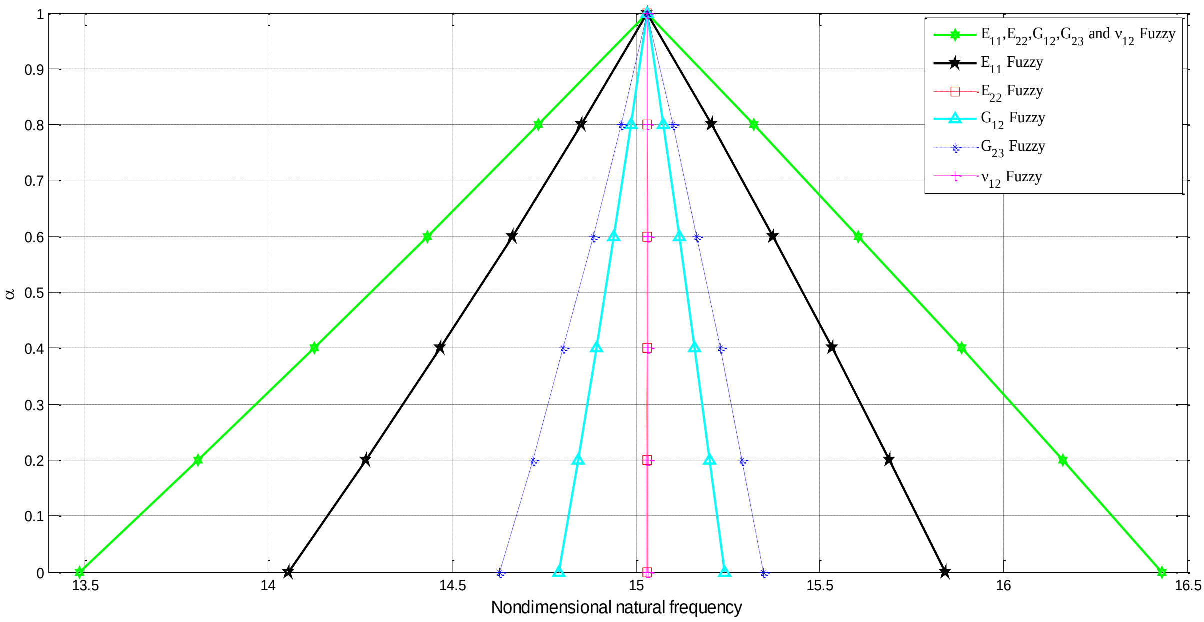


Figure 5.11 Nondimensional natural frequency distribution of for a/h=10, [$0^0/90^0/90^0/0^0$], Material-II

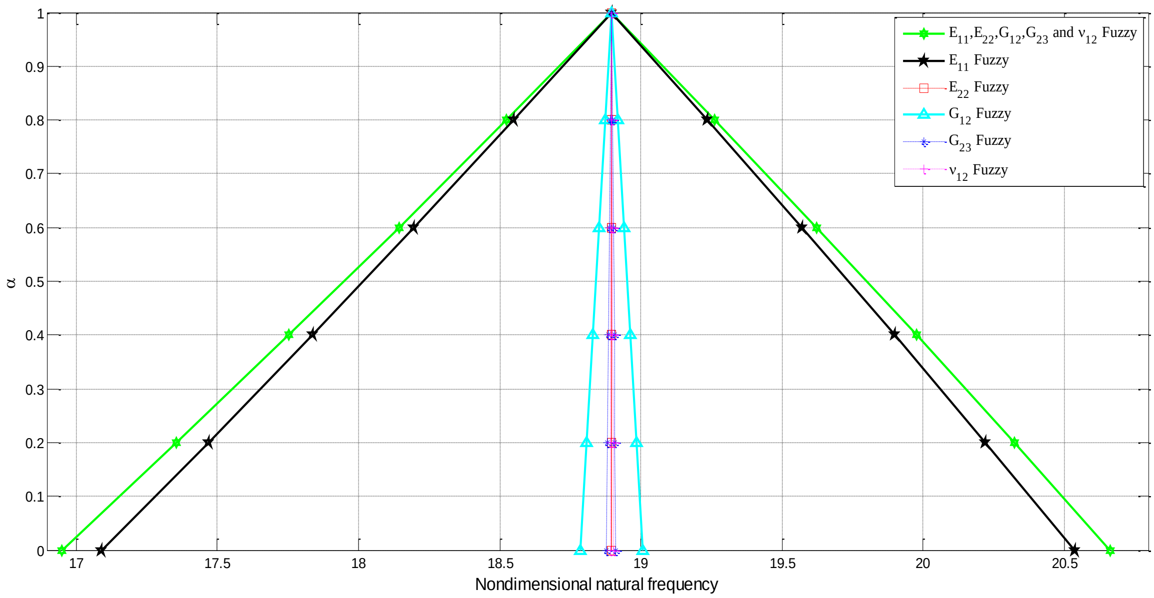


Figure 5.12 Nondimensional natural frequency distribution for a/h=100, [$0^0/90^0/90^0/0^0$], Material-II

It is well known that the stability and stiffness of composite structures largly depends on the aspect ratio and thickness ratio hence, it is essential to know the effect of thickness ratio on nondimensional natural frequency. Fig. 5.5-5.12 shows the nondimensional natural frequency distribution of 2-layers cross ply and 4-layers symmetric cross-ply laminated composite plate for two different thickness ratios (*a/h*=10 and 100). It is entirely clear from the figures that the nondimensional natural frequency values are increasing as thickness ratio increases.

The performance of the laminate is depended on stiffness, which depends upon the material properties and the laminate material property greatly depends on the lamination scheme, therefore studying the effect of lamination scheme on the nondimensional natural frequency is essential. From the result is shown in Fig. 5.5-5.12, it can be observed that the nondimensional natural frequency values are increasing as the number of layers due to the change in stiffness property of the laminate as discussed earlier.

Overall performance of the composite structure depends upon the material properties of the laminates. The nondimensional natural frequency distribution for the laminated composite plate presented in Fig. 5.5-5.12. It can be seen that even small change in the E_{11} will result into substantial changes in the nondimensional natural frequency. The contribution of E_{22} is very minimal, the second prominent role-playing material properties parameter is G_{12} and G_{23}. Where all material properties parameter E_{11}, E_{22}, G_{12}, G_{23} and ν_{12} are fuzzy, the largest possible distribution range is obtained. The smallest possible distribution range received where ν_{12} and E_{22} are fuzzy.

5.4 Bending Analysis of The Laminated Composite Plate

In this section, laminated composite plate with uncertain material properties is considered for the static analysis. Initially, the validation of developed code is performed by comparing the results with those available in the literature. Further, a detailed parametric study was performed for the laminated composite plate, in order to check the efficiency of the present numerical model for static analysis, and the results obtained are presented and discussed.

The problem of laminated composite plates under transverse loading is considered. The framework of HSDT is used to calculate the central deflection of the composite plate. The randomness in material properties is introduced using the fuzzy membership function as described in Chapter 3. The geometrical and material properties taken from (Zhang and Kim, 2006) for the central deflection calculation and are as follows:

Length (L) =12 inch, thickness (t) =0.096 inch,

E_1 =1.8282 x 10^6 Psi, E_2 =1.8315 x 10^6 Psi, G_{12}= G_{13}=G_{23}=3.125 x 10^6 Psi and ν_{12}= 0.023949.

The laminated composite plate is subjected to a uniformly distributed load of 0.4, 0.8, 1.2, 1.6, 2.0 Psi. Here again uncertainty in the material properties like the longitudinal modulus, transverse modulus, shear modulus and poisons ratio are considered in the range of [±20%] about their deterministic values.

5.4.1 Convergence and validation study for central deflection of the laminated composite plate

The convergence behavior of the developed mathematical model has checked for the central deflection and presented in Fig. 5.13.

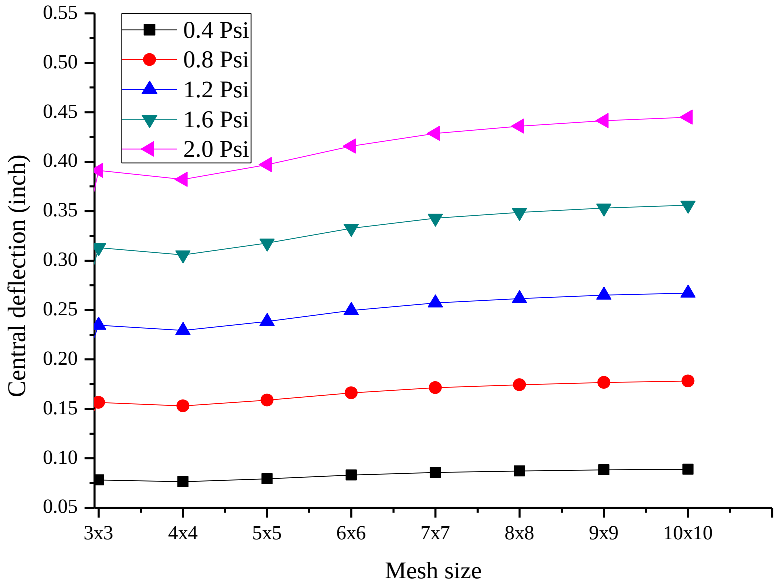


Figure 5.13 Convergence study of central deflection of laminated composite plate

Based on the convergence study it is supposed that an (8×8) mesh size is suitable for computation, hence an (8×8) mesh size used for computing the central deflection of the laminated composite plate. The validation of the proposed technique that is based on fuzzy set theory is performed by computing the central deflection of the laminated composite plate with fuzzified material properties. The computed results with their deterministic values (at α=1) compared with the that are available from other deterministic analysis using FEM as in the reference (Zhang and Kim, 2006) and are presented in Table 5.3.

Table 5.3 Comparison study of central deflection of cantilever laminated composite plate with uniformly distributed load

Load (Psi)	**Central deflection (inch)**	
	FFEM (α=1)	(Zhang and Kim, 2006)
0.4	0.0872	0.0842
0.8	0.1744	0.1684
1.2	0.2616	0.2525
1.6	0.3488	0.3367
2.0	0.436	0.4209

It can be observed from the Table 5.3 that the crisp value of the central deflection of the laminated composite plate is good agreement with the value obtained from FFEA with α=1, which validates the solution accuracy of the present model.

To know the effect of variation in the material properties on the deflection, a numerical study carried out by changing the stacking sequence and thickness ratio of the laminated composite plate. Geometric and material properties are taken as follows.

Length (L) =0.3048 m, thickness (t) =0.002438 m,

E_{11} =12.605 Gpa, E_{22} =12.6277 Gpa, G_{12}= 2.1546 Gpa and v_{12}= 0.23949. Plate is subjected to uniformly distributed load of 2757.9 N.

The uncertainty in the material properties is considered in the range of [±20%] about their deterministic values. Triangular membership function used for modeling the uncertain material properties are presented in Fig. 5.14 and the computed results are presented in Fig. 5.15-5.18.

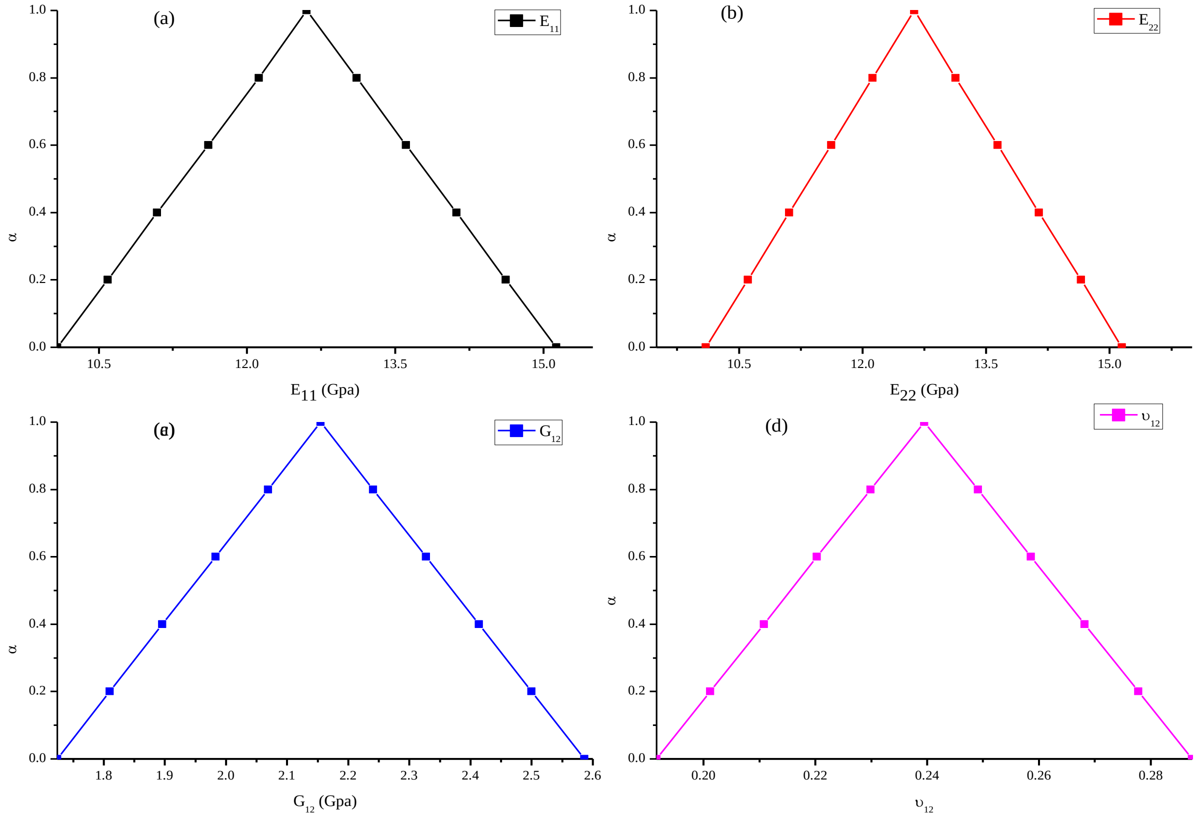


Figure 5.14 Membership function of the material properties

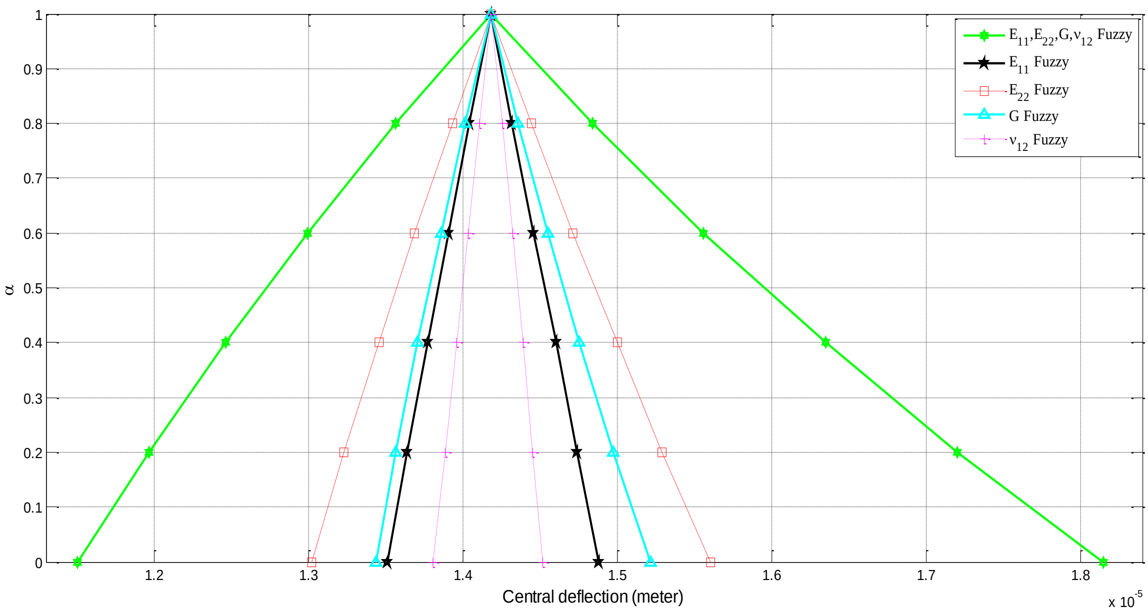


Figure 5.15 Central deflection distribution for a/h=10, [$0^0/90^0/0^0/90^0$]

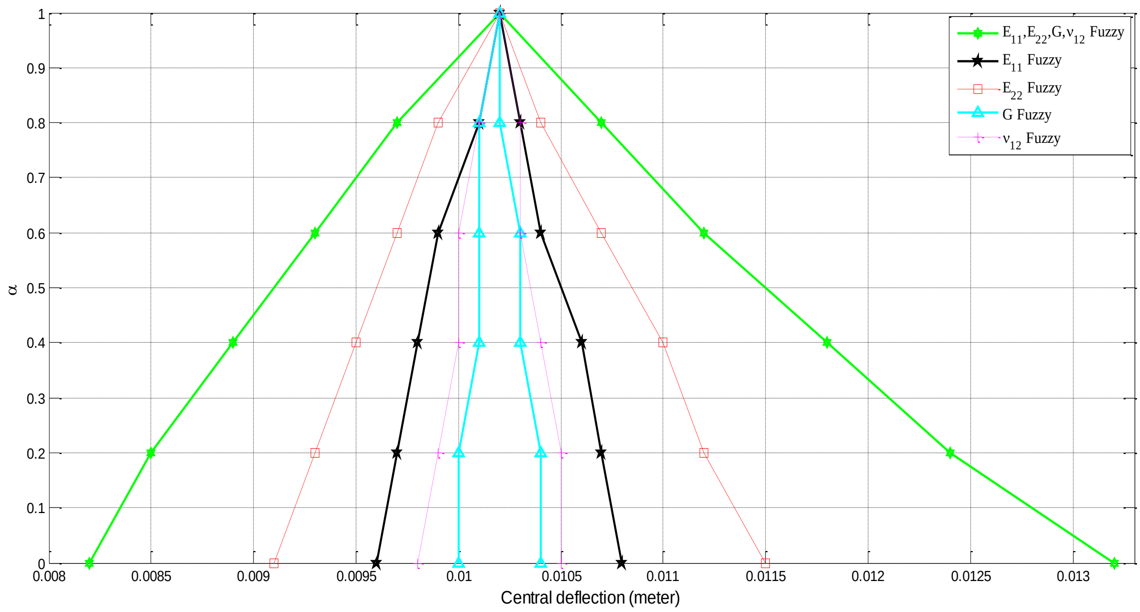


Figure 5.16 Central deflection distribution for a/h=100, [$0^0/90^0/0^0/90^0$]

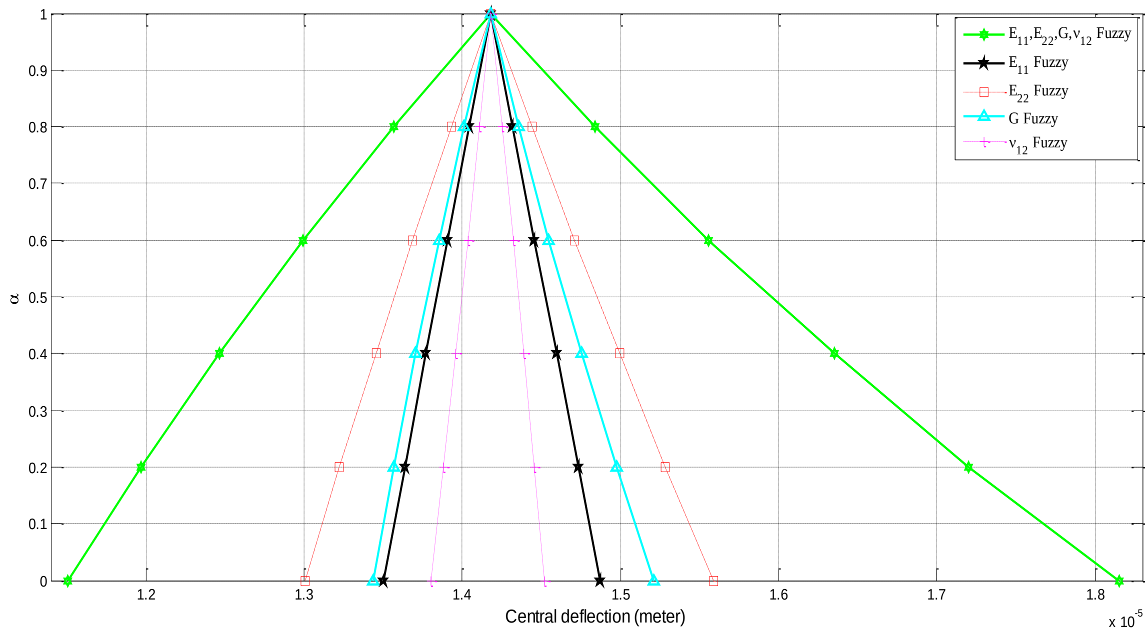


Figure 5.17 Central deflection distribution for a/h=10, [$0^0/90^0/0^0/90^0/0^0/90^0/0^0/90^0$]

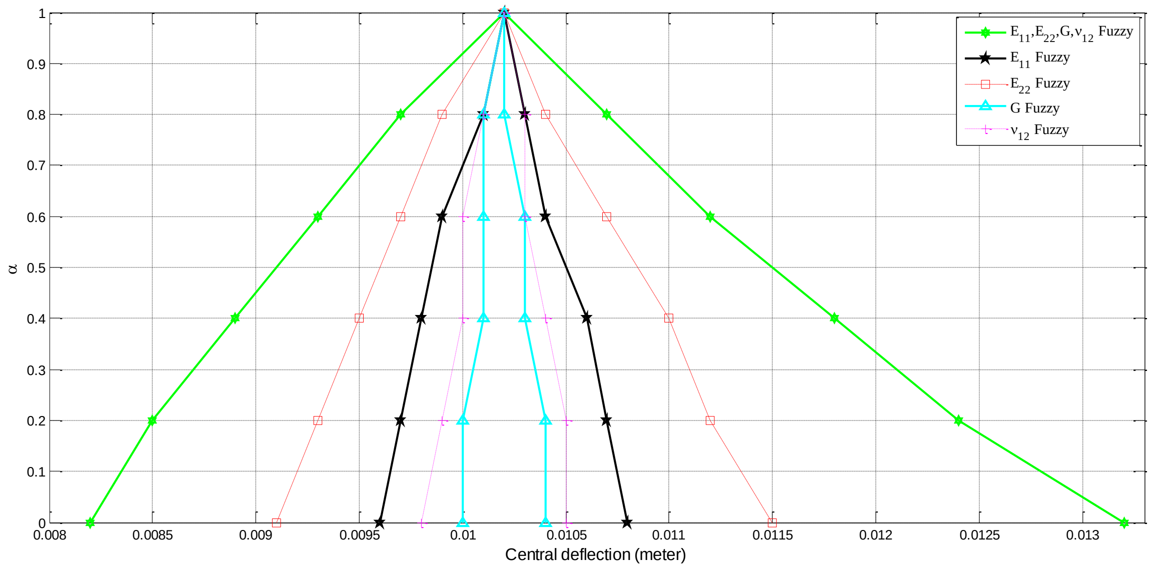


Figure 5.18 Central deflection distribution for a/h=100, [$0^0/90^0/0^0/90^0/0^0/90^0/0^0/90^0$]

Figure 5.15-5.18 shows the distribution of the central deflection of a laminated composite plate of four layers and eight layers composite plates for two different thickness ratios (*a/h*=10 and 100). It is entirely clear from the figures that the central deflection values are increasing as thickness ratio increases.

The lamination scheme plays a vital role in the performance of structures. From the result presented in Fig. 5.15-5.18, it is seen that the central deflection values are increasing only for the variation in the material properties greater than fifteen percent. If the variation in material properties is limited to ten percent from crisp values, change in the central deflection is very minimal.

Also, it can be seen that change in the value of material properties result in a change in central deflection of the composite plate. The small change in the E_{22} will result in significant contribution in the changes in the central deflection for the all 4 cases, which shows that E_{22} plays a prominent role. The second noticeable role-playing material properties parameter is G for thickness ratio a/h=10 and E_{11} for thickness ratio a/h=100. When all material properties E_{11}, E_{22}, G and ν_{12} are varying, the largest possible distribution range for central deflection for the plate is obtained. The smallest possible distribution range obtained where only ν_{12} is fuzzy for thickness ratio a/h=10 and where G is fuzzy, for the thickness ratio a/h=100.

5.5 Conclusions

Based on the convergence, validation and numerical study presented in this chapter, the following conclusion can be drawn:

1. The parametric study indicates that the natural frequency and static deflection distribution greatly dependent on the composite material properties and the type of laminated scheme (symmetric or anti-symmetric and cross-ply).
2. Natural frequency is higher for the symmetrical cross-ply and angle-ply laminate in comparison to anti-symmetric laminates.
3. The increase in thickness ratio of the plate results in a reduction in the nondimensional frequency distribution.

Chapter- 6

GENERAL OBSERVATIONS AND CONCLUSIONS

6.1 Introduction

In chapters one to five, we have discussed the importance of the FFEM method. The related work is presented, and the general mathematical formulation is proposed for the laminated composite plate. Specific examples with the results for the laminated composite plate related to free vibration and static deflection have discussed. In the present chapter, we summarize some of the general observations and conclusions.

6.2 Conclusions

From the results discussed in the previous chapters, it can be seen that the effect of uncertainty on the response of laminated composites plate is a complex phenomenon. It is dependent on a large number of parameters, one of them is material property uncertainty. The general conclusions that can be derived from the fuzzy finite element analysis of the laminated composite plate are as follows:

1. The FFEM gives more accurate results for the composite plate with uncertain material properties. For the range of variation in the input material properties, triangular fuzzy membership function was considered.
2. Lower range of variation in the material properties, the change in the fiber orientation, aspect ratio, do not have much influence on the response of the laminated composite plate.
3. The nature and the influence of the each one of the material property parameter on the response distribution are different.
4. For the case of cross-ply laminate, a variation of E_{11} has the maximum influence on the response distribution of the structure.
5. If the other parameter remains unchanged, the distribution of deflection is lower, and distribution of the natural frequency is higher for stiffer laminate configuration.

6.3 Significant Contribution of The Present Work

The present work is an effort to fill the knowledge gap, for the uncertainty quantification in the domain of the laminated composite plate by using FFEM and are as follows:

1. A general mathematical model of laminated composite plates has been developed in the framework of the HSDT, considering all the higher order terms in the mathematical formulation,which will accurately predict the behaviour of laminated composite plate with material property uncertainty.

2. A fuzzy-based finite element analysis is proposed using a nine noded isoperimetric element having nine degrees of freedom per node for the discretization of the laminated composite plate model.

3. A home-made computer program developed in the MATLAB platform for obtaining the structure response distribution having material uncertainty by integrating fuzzy arithmetic with the finite element steps.

4. Diverse numerical problem have studied to show the efficacy of the present developed mathematical and simulation model supported by the convergence and validation study of the natural frequency and static deflection behaviour of the laminated plate.

It is understood from the discussions that the developed mathematical composite plate model in conjunction with HSDT would be useful for the analysis of the laminated composite plate having uncertainty in the system parameter. It is worth to mention that, with the introduction of fuzzy concept in conjuction with the FEM, FEEM is capable to generate the natural frequency and static deflection response distribution of the laminated composite structure considering uncertainty in the design parameter with greater accuracy and very less computational time.

6.3 Future Scope of The Research

The following problems can be attempted as an extension of the present work:

1. The problem with uncertainty in applied static or dynamic loading can be studied.
2. The problems where geometric parameters and boundary conditions are uncertain.
3. The problem with the nonlinear analysis of laminated composite plate with the uncertainty in the design parameter.

REFERENCES

Adhikari, S., and H. Haddad Khodaparast. 2014. "A Spectral Approach for Fuzzy Uncertainty Propagation in Finite Element Analysis." *Fuzzy Sets and Systems* 243. Elsevier B.V.: 1–24. doi:10.1016/j.fss.2013.10.005.

Akpan, U.O., T.S. Koko, I.R. Orisamolu, and B.K. Gallant. 2001. "Practical Fuzzy Finite Element Analysis of Structures." *Finite Elements in Analysis and Design* 38 (2): 93–111. doi:10.1016/S0168-874X(01)00052-X.

Balu, A.S., and B.N. Rao. 2012. "High Dimensional Model Representation Based Formulations for Fuzzy Finite Element Analysis of Structures." *Finite Elements in Analysis and Design* 50 (March): 217–30. doi:10.1016/j.finel.2011.09.012.

Behera, Diptiranjan, and Chakraverty S. 2013. "Fuzzy Finite Element Based Solution of Uncertain Static Problems of Structural Mechanics." *International Journal of Computer Applications* 69 (15): 6–11.

Chen, Li, and S.S. Rao. 1997. "Fuzzy Finite-Element Approach for the Vibration Analysis of Imprecisely-Defined Systems." *Finite Elements in Analysis and Design* 27 (1): 69–83. doi:10.1016/S0168-874X(97)00005-X.

Cherki, A., G. Plessis, B. Lallemand, T. Tison, and P. Level. 2000. "Fuzzy Behavior of Mechanical Systems with Uncertain Boundary Conditions." *Computer Methods in Applied Mechanics and Engineering* 189 (3): 863–73. doi:10.1016/S0045-7825(99)00401-6.

Cook, Robert D., Malkus, Plesha, and Witt. 2007. *Concepts and Applicatioins of Finite Element Analysis , 4th Edition*. Wiley India Pvt. Limited.

Dash, Padmanav, and B. N. Singh. 2015. "Static Response of Geometrically Nonlinear Laminated Composite Plates Having Uncertain Material Properties." *Mechanics of Advanced Materials and Structures* 22: 269–80. doi:10.1115/1.4006757.

Dhingra, Anoop K., Singiresu S. Rao, and Virendra Kumar. 1992. "Nonlinear Membership Functions in Multiobjective Fuzzy Optimization of Mechanical and Structural Systems." *AIAA Journal* 30 (1): 251–60. doi:10.2514/3.10906.

Elishakoff, Isaac. 1998. "Three Versions of the Finite Element Method Based on Concepts of Either Stochasticity, Fuzziness, or Anti-Optimization." *Applied Mechanics Reviews* 51 (2): 209–18.

Giannini, Oliviero, and Michael Hanss. 2008. "The Component Mode Transformation Method: A Fast Implementation of Fuzzy Arithmetic for Uncertainty Management in Structural Dynamics." *Journal of Sound and Vibration* 311 (3-5): 1340–57. doi:10.1016/j.jsv.2007.10.029.

Guang-yuan, Wang, and Ou Jin-ping. 1990. “Theory of Fuzzy Random Vibration with Fuzzy Parameters.” *Fuzzy Sets and Systems* 36: 103–12.

Lal, Achchhe, and B.N. Singh. 2011. “Effect of Random System Properties on Bending Response of Thermo-Mechanically Loaded Laminated Composite Plates.” *Applied Mathematical Modelling* 35 (12): 5618–35. doi:10.1016/j.apm.2011.05.014.

Lallemand, B., G. Plessis, T. Tison, and P. Level. 1999. “Neuman Expansion for Fuzzy Finite Element Analysis.” *Engineering Computations* 16 (5): 572–83.

Liu, Qing, and Singiresu S Rao. 2003. “A Fuzzy Approach to Fiber-Reinforced Composite Material Mechanics.” In *44th AIAA Structures, Structural Dynamics and Material Conference 7-10 April*, 1–10.

Liu, Qing, and Singiresu S. Rao. 2005. “Fuzzy Finite Element Approach for Analysis of Fiber-Reinforced Laminated Composite Beams.” *AIAA Journal* 43 (3): 651–61. doi:10.2514/1.940.

MacE, Brian R., D. V H Vandepitte, and Pascal Lardeur. 2005. “Uncertainty in Structural Dynamics.” *Finite Elements in Analysis and Design* 47 (1): 1–3. doi:10.1016/j.finel.2010.07.021.

Massa, F., T. Tison, and B. Lallemand. 2009. “Fuzzy Modal Analysis: Prediction of Experimental Behaviours.” *Journal of Sound and Vibration* 322 (1-2): 135–54. doi:10.1016/j.jsv.2008.10.032.

Massa, Franck, T. Tison, and B. Lallemand. 2006. “A Fuzzy Procedure for the Static Design of Imprecise Structures.” *Computer Methods in Applied Mechanics and Engineering* 195 (9-12): 925–41. doi:10.1016/j.cma.2005.02.015.

Moens, D, and D. Vandepitte. 2006. “Recent Advances in Non-Probabilistic Approaches for Non-Deterministic Dynamic Finite Element Analysis.” *Archives of Computational Methods in Engineering* 13 (3): 389–464. doi:10.1007/BF02736398.

Moens, D., and D Vandepitte. 2005. “A Fuzzy Finite Element Procedure for the Calculation of Uncertain Frequency-Response Functions of Damped Structures: Part 2 - Procedure.” *Journal of Sound and Vibration* 288 (3): 431–62. doi:10.1016/j.jsv.2005.07.001.

Moens, D., and D. Vandepitte. 2002. “Fuzzy Finite Element Method for Frequency Response Function Analysis of Uncertain Structures.” *AIAA Journal* 40 (1): 126–36. doi:10.2514/3.15005.

Moens, David, and Michael Hanss. 2011. “Non-Probabilistic Finite Element Analysis for Parametric Uncertainty Treatment in Applied Mechanics: Recent Advances.” *Finite Elements in Analysis and Design* 47 (1). Elsevier: 4–16. doi:10.1016/j.finel.2010.07.010.

Moore, R E, Baker R, and Cloud M. 2009. *Introduction to Interval Analysis*. *Mathematics of Computation*. Vol. 22. doi:10.2307/2004792.

Muhanna, Rafi L., and Robert L. Mullen. 1999. “Formulation of Fuzzy Finite-Element Methods for Solid Mechanics Problems.” *Computer-Aided Civil and Infrastructure Engineering* 14 (2): 107–17. doi:10.1111/0885-9507.00134.

Nakagiri, S, Takabatake H., and S. Tani. 1987. “Uncertain Eigenvalue Analysis of Composite Laminated Plate by Stochastic Finite Element Method.” *Journal of Manufacturing Science and Engineering* 109 (1): 9–12. doi:10.1299/kikaia.51.1510.

Noor, Ahmed K., James H. Starnes, and Jeanne M. Peters. 2000. “Uncertainty Analysis of Composite Structures.” *Computer Methods in Applied Mechanics and Engineering* 185 (2-4): 413–32. doi:10.1016/S0045-7825(99)00269-8.

Onkar, Amit Kumar, and D. Yadav. 2005. “Forced Nonlinear Vibration of Laminated Composite Plates with Random Material Properties.” *Composite Structures* 70 (3): 334–42. doi:10.1016/j.compstruct.2004.08.037.

Pawar, Prashant M., Sung Nam Jung, and Babruvahan P. Ronge. 2012. “Fuzzy Approach for Uncertainty Analysis of Thin Walled Composite Beams.” *Aircraft Engineering and Aerospace Technology* 84 (1): 13–22. doi:10.1108/00022661211194942.

Rama Rao, M. V., and R. Ramesh Reddy. 2007. “Analysis of a Cable-Stayed Bridge with Multiple Uncertainties - A Fuzzy Finite Element Approach.” *Journal of Structural Engineering* 33 (6): 523–25.

Rao, S. S., and L. Chen. 1998. “Numerical Solution of Fuzzy Linear Equations in Engineering Analysis.” *International Journal for Numerical Methods in Engineering* 43 (3): 391–408. http://dx.doi.org/10.1002/(SICI)1097-0207(19981015)43:3<391::AID-NME417>3.0.CO;2-J.

Rao, S.S., and Sawyer James P. 1995. “Fuzzy Finite Element Approach for the Analysis of Imprecisely Defined Systems.” *AIAA Journal* 33 (12): 2364–70.

Reddy, J. N. 2003. *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis*. Second Edi. CRC Press Book.

Salim, S., D. Yadav, and N.G.R. Iyengar. 1993. “Analysis of Composite Plates with Random Material Characteristics.” *Mechanics Research Communications* 20 (5): 405–14.

Shaker, Afeefa, Wael G. Abdelrahman, Mohammad Tawfik, and Edward Sadek. 2008. “Stochastic Finite Element Analysis of the Free Vibration of Laminated Composite Plates.” *Computational Mechanics* 41 (4): 493–501. doi:10.1007/s00466-007-0205-7.

Simoen, Ellen, Guido De Roeck, and Geert Lombaert. 2015. “Dealing with Uncertainty in Model Updating for Damage Assessment: A Review.” *Mechanical Systems and Signal Processing* 56-57. Elsevier: 123–49. doi:10.1016/j.ymssp.2014.11.001.

Singh, B. N., D. Yadav, and N. G. R. Iyengar. 2002. “A C° Element for Free Vibration of Composite Plates with Uncertain Material Properties.” *Advanced Composite*

Materials 11 (4). Taylor & Francis Group: 331–50. doi:10.1163/156855102321669163.

Singh, B.N., and Grover Neeraj. 2013. “Stochastic Methods for the Analysis of Uncertain Composites.” *Journal of the Indian Institute of Science* 93 (4): 603–19.

Singh, B.N., D. Yadav, and N.G.R. Iyengar. 2001. “Natural Frequencies of Composite Plates with Random Material Properties Using Higher-Order Shear Deformation Theory.” *International Journal of Mechanical Sciences* 43 (10): 2193–2214. doi:10.1016/S0020-7403(01)00046-7.

Stefanou, George. 2009. “The Stochastic Finite Element Method: Past, Present and Future.” *Computer Methods in Applied Mechanics and Engineering* 198 (9-12): 1031–51. doi:10.1016/j.cma.2008.11.007.

Van Vinckenroy, G., and W.P. de Wilde. 1995. “The Use of Monte Carlo Techniques in Statistical Finite Element Methods for the Determination of the Structural Behaviour of Composite Materials Structural Components.” *Composite Structures* 32 (1-4): 247–53. doi:10.1016/0263-8223(95)00055-0.

Vanmarcke, Erik, and Mircea Grigoriu. 1983. “Stochastic Finite Element Analysis of Simple Beams.” *Journal of Engineering Mechanics* 109 (5). American Society of Civil Engineers: 1203–14. doi:10.1061/(ASCE)0733-9399(1983)109:5(1203).

Wood, Kristin L., Erik K. Antonsson, and James L. Beck. 1990. “Representing Imprecision in Engineering Design: Comparing Fuzzy and Probability Calculus.” *Research in Engineering Design* 1 (3-4): 187–203. doi:10.1007/BF01581211.

Xia, Y, and M I Friswell. 2014. “Efficient Solution of the Fuzzy Eigenvalue Problem in Structural Dynamics.” *Engineering Computations* 31 (5): 864–78. doi:10.1108/EC-02-2013-0052.

Yao, Wen, Xiaoqian Chen, Wencai Luo, Michel Van Tooren, and Jian Guo. 2011. “Review of Uncertainty-Based Multidisciplinary Design Optimization Methods for Aerospace Vehicles.” *Progress in Aerospace Sciences* 47 (6). Elsevier: 450–79. doi:10.1016/j.paerosci.2011.05.001.

Zadeh, L.a. 1965. “Fuzzy Sets.” *Information and Control* 8 (3): 338–53. doi:10.1016/S0019-9958(65)90241-X.

Zhang, Y. X., and K. S. Kim. 2006. “Geometrically Nonlinear Analysis of Laminated Composite Plates by Two New Displacement-Based Quadrilateral Plate Elements.” *Composite Structures* 72 (3): 301–10. doi:10.1016/j.compstruct.2005.01.001.